Kinetic gases as sources of gravity

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Center of Excellence “The Dark Side of the Universe”

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Theoretical Physics Seminar

Outline

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Problems in gravity and cosmology

- So far unexplained cosmological observations:
  - Accelerating expansion of the universe
  - Homogeneity of cosmic microwave background

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- Idea here: modification of the geometrical structure of spacetime!
  - Replace metric spacetime geometry by Finsler geometry.
  - Similarly: replacing flat spacetime by curved spacetime led to GR.
  - Replace perfect fluid model by velocity-dependent distribution of particles.

Fluids are everywhere

- Perfect fluid:
  - No shear stress, no friction.
  - Characterized by density ρ and pressure p.
    - ⋆ Dust, dark matter: $p = 0$.
    - ⋆ Radiation: $p = \frac{1}{3}\rho$.
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- Charged, multi-component gas:
  - Plasma, interacting gas including recombination / ionization.
  - Used in stellar dynamics, pre-CMB era models...

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  - Approaches to quantum gravity
  - Electrodynamics in anisotropic media
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  - Other matter field theories
- Possible explanations of yet unexplained phenomena:
  - Fly-by anomaly
  - Galaxy rotation curves
  - Accelerating expansion of the universe
  - Inflation

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The clock postulate

- Proper time along a curve in Lorentzian spacetime:

$$\tau = \int_{t_1}^{t_2} \sqrt{-g_{ab}(x(t))\dot{x}^a(t)\dot{x}^b(t)}dt\,.$$

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- Finsler geometry: use a more general length functional:

$$\tau = \int_{t_1}^{t_2} F(x(t), \dot{x}(t))\,dt\,.$$

- Finsler function $F : TM \to \mathbb{R}^+$.
- Parametrization invariance requires homogeneity:

$$F(x, \lambda y) = \lambda F(x, y) \quad \forall \lambda > 0\,.$$

Finsler spacetime geometry

- Finsler geometries suitable for spacetimes exist. [C. Pfeifer, M. Wohlfarth '11]

⇒ Finsler metric with Lorentz signature:

$$g^F_{ab}(x,y) = \frac{1}{2}\frac{\partial}{\partial y^a}\frac{\partial}{\partial y^b}F^2(x,y)\,.$$

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- Unit vectors $y \in T_xM$ defined by

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⇒ Set $\Omega_x \subset T_xM$ of unit timelike vectors at $x \in M$.

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⇒ Set $\Omega_x \subset T_xM$ of unit timelike vectors at $x \in M$.

- Ω_x contains a closed connected component $S_x \subseteq \Omega_x$.

⇝ Causality: S_x corresponds to physical observers.

Geometry on the tangent bundle

- Cartan non-linear connection:

$$N^a{}_b = \frac{1}{4}\bar{\partial}_b\left[g^{F\,ac}(y^d\partial_d\bar{\partial}_c F^2 - \partial_c F^2)\right]$$

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⇒ Split of the tangent and cotangent bundles:

- Tangent bundle: $TTM = HTM \oplus VTM$

$$\delta_a = \partial_a - N^b{}_a\bar{\partial}_b\,, \quad \bar{\partial}_a$$

- Cotangent bundle: $T^*TM = H^*TM \oplus V^*TM$

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$$G = -g^F_{ab}\,dx^a \otimes dx^b - \frac{g^F_{ab}}{F^2}\,\delta y^a \otimes \delta y^b$$

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- Geodesic spray:

$$\mathbf{S} = y^a\delta_a$$

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- Geometric structures defined on observer space:
  - Pullback of Hilbert form $\omega = \bar{\partial}_a F\, dx^a$ to O.
  - Sasaki metric $\tilde{G}$ on O given by pullback of G to O.
  - Volume form Σ of Sasaki metric $\tilde{G}$:

$$\Sigma = \frac{1}{3!}\omega \wedge d\omega \wedge d\omega \wedge d\omega\,.$$

  - Geodesic spray $\mathbf{S}$ is tangent to O; restricts to Reeb vector field $\mathbf{r} = \mathbf{S}|_O$.
  - Geodesic hypersurface measure:

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- Further useful relations: $\mathcal{L}_{\mathbf{r}}\omega = 0$; $\mathcal{L}_{\mathbf{r}}\Sigma = 0$; $\mathcal{L}_{\mathbf{r}}\Omega = 0$; $d\Omega = 0$; $\Sigma = \omega \wedge \Omega$; $\iota_{\mathbf{r}}\omega = 1$.

From metric to Finsler geometry

- Tangent bundle geometry:
  - Finsler function:
$$F(x,y) = \sqrt{|g_{ab}(x)y^a y^b|}$$
  - Finsler metric:
$$g^F_{ab}(x,y) = \begin{cases} -g_{ab}(x) & y \text{ timelike} \\ g_{ab}(x) & y \text{ spacelike} \end{cases}$$
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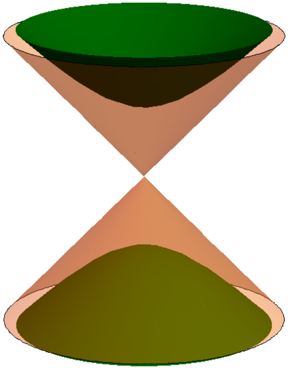
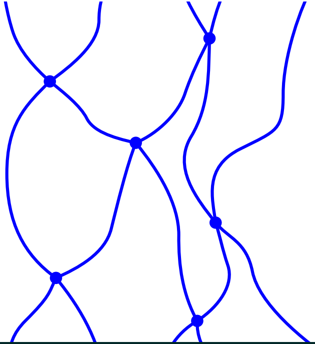


- Observer space:
  - Space Ω_x of unit timelike vectors at $x \in M$.
  - Space S_x of future unit timelike vectors at $x \in M$.
  - Observer space O: union of shells S_x.

Outline

Definition of fluids

- Single-component fluid:
  - Constituted by classical, relativistic particles.
  - Particles have equal properties (mass, charge, . . .).
  - Particles follow piecewise geodesic curves.
  - Endpoints of geodesics are interactions with other particles.



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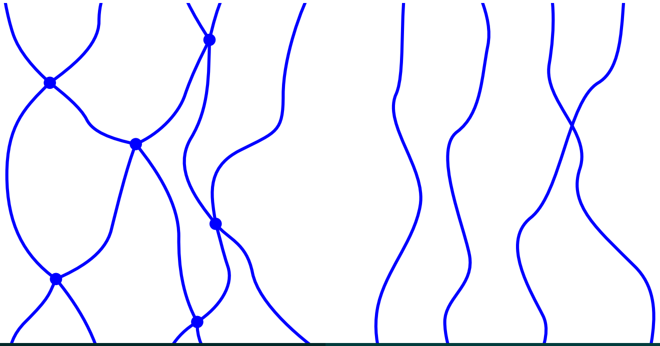

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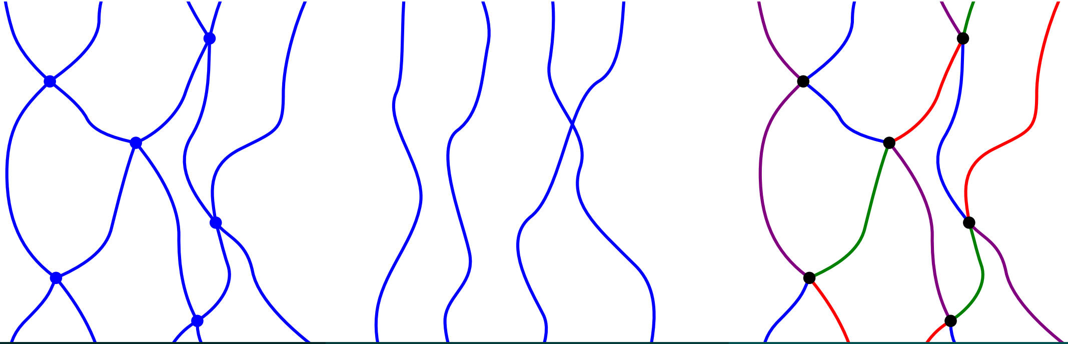
- Multi-component fluid: multiple types of particles.



Geodesics on observer space

- Dynamics of fluids depends on geodesic equation.
- Geodesic equation for curve $x(\tau)$ on spacetime M:

$$\ddot{x}^a + N^a{}_b(x, \dot{x})\dot{x}^b = 0\,.$$

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- Canonical lift of curve to tangent bundle TM:

$$x\,, \quad y = \dot{x}\,.$$

- Lift of geodesic equation:

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- Tangent vectors are future unit timelike: $(x, y) \in O$.

⇒ Particle trajectories are piecewise integral curves of **r** on O.

One-particle distribution function

- Recall: $\Omega = \iota_{\mathbf{r}}\Sigma \in \Omega^6(O)$ unique 6-form such that:
  - Ω non-degenerate on every hypersurface not tangent to $\mathbf{r}$.
  - $d\Omega = 0$.

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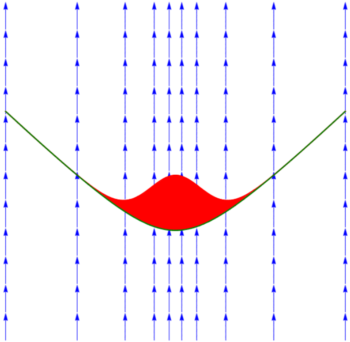
- Define one-particle distribution function $\phi : O \to \mathbb{R}^+$ such that:

For every hypersurface $\sigma \subset O$,

$$N[\sigma] = \int_{\sigma} \phi\Omega$$

# of particle trajectories through σ.



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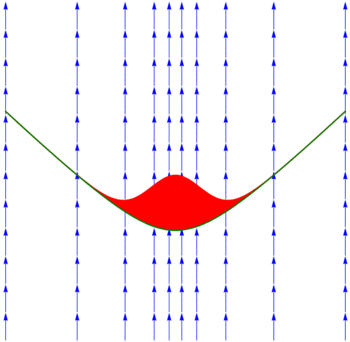

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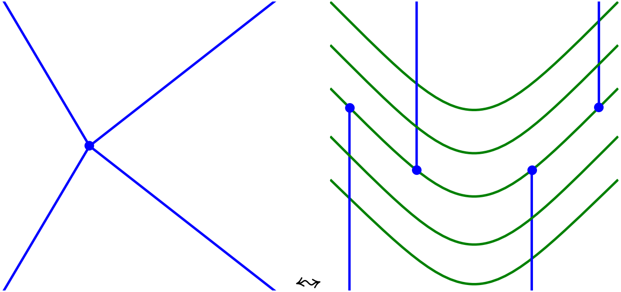
  - Counting of particle trajectories respects hypersurface orientation.
- For multi-component fluids: ϕ_i for each component i.

Collisions & the Liouville equation

- Collision in spacetime $\leftrightsquigarrow$ interruption in observer space.



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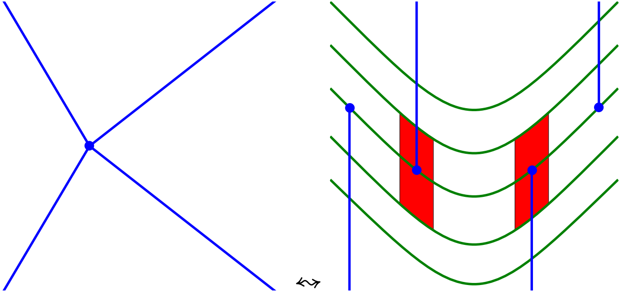

- For any open set $V \in O$,

$$\int_{\partial V} \phi\Omega = \int_V d(\phi\Omega) = \int_V \mathcal{L}_{\mathbf{r}}\phi\Sigma$$

# of outbound trajectories - # of inbound trajectories.

$\Rightarrow$ Collision density measured by $\mathcal{L}_{\mathbf{r}}\phi$.

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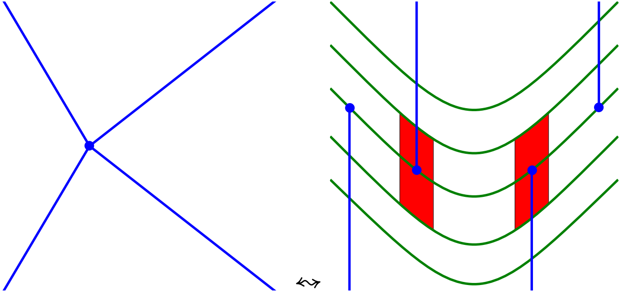

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$\Rightarrow$ Collision density measured by $\mathcal{L}_{\mathbf{r}}\phi$.

- Collisionless fluid: trajectories have no endpoints, $\mathcal{L}_{\mathbf{r}}\phi = 0$.

$\Rightarrow$ Simple, first order equation of motion for collisionless fluid.

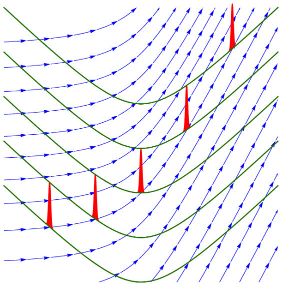
$\Rightarrow$ ϕ is constant along integral curves of $\mathbf{r}$.

Outline

Some (very) pictorial examples

Geodesic dust fluid:

$$\phi(x,y) \sim \delta(y - u(x)) .$$



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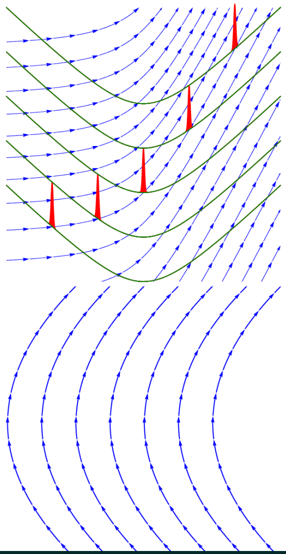

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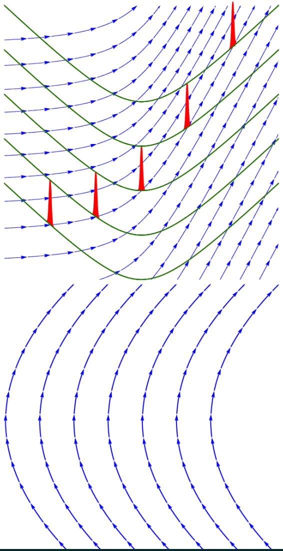
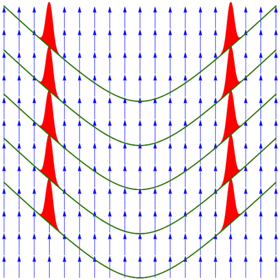


Collisionless fluid:

$$\mathcal{L}_{\mathbf{r}}\phi = 0 .$$



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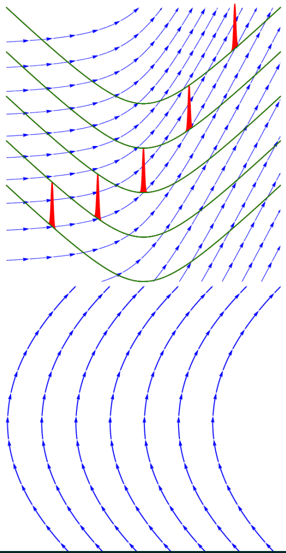

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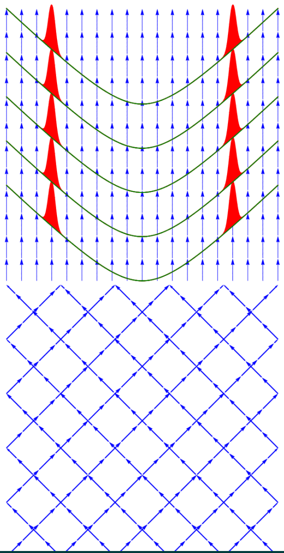

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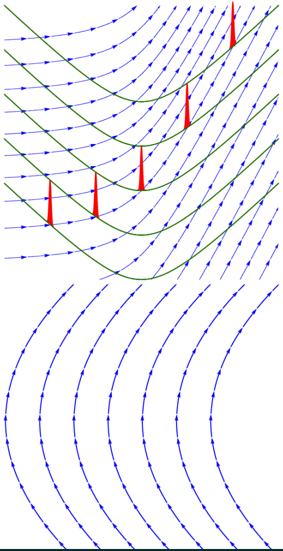

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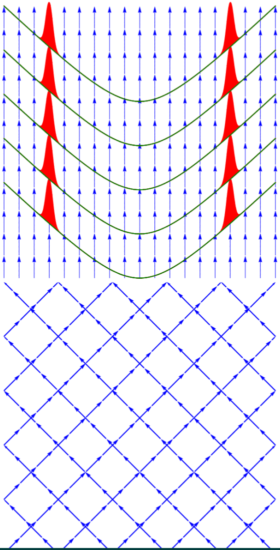
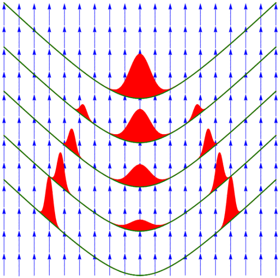


Interacting fluid:

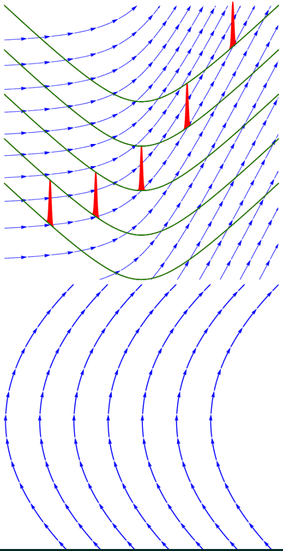
$$\mathcal{L}_{\mathbf{r}}\phi \neq 0 .$$



Some (very) pictorial examples

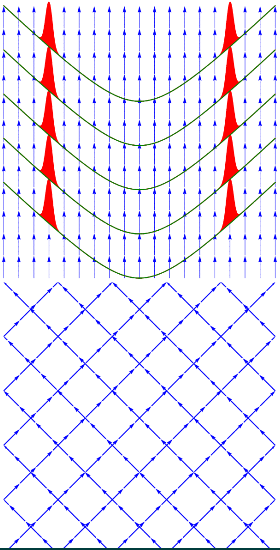
Geodesic dust fluid:

$$\phi(x,y) \sim \delta(y - u(x))\,.$$



Collisionless fluid:

$$\mathcal{L}_{\mathbf{r}}\phi = 0\,.$$



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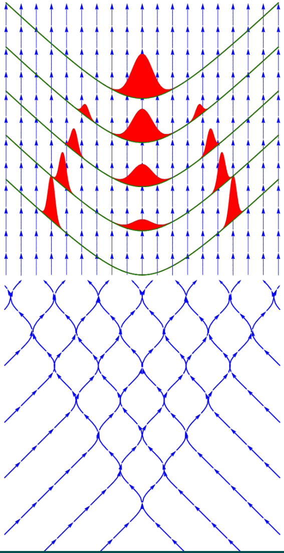

Example: collisionless dust fluid

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  - Mass density $\rho : M \to \mathbb{R}^+$.
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⇒ Generalized (pressureless) Euler equations to Finsler geometry [MH '15].

- Metric limit $F^2(x,y) = |g_{ab}(x)y^a y^b|$ yields Euler equations:

$$u^b \nabla_b u^a = 0 , \quad \nabla_a(\rho u^a) = 0 .$$

Cosmological symmetry

- Introduce suitable coordinates on TM:

$$t, r, \theta, \varphi, y^t, y^r, y^\theta, y^\varphi\,.$$

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- Introduce new coordinates: $\tilde{y} = y^t \tilde{F}(t, w/y^t)$, $\tilde{w} = w/y^t$.

⇒ Coordinates on observer space O with $\tilde{y} \equiv 1$.

⇒ Geometry function $\tilde{F}(t, \tilde{w})$ on O.

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- Example: collisionless dust fluid $\phi(x, y) \sim \rho(x)\delta_{S_x}(y, u(x))$:

$$u(t) = \frac{1}{\tilde{F}(t, 0)}\partial_t\,, \quad \partial_t\left(\rho(t)\sqrt{g^F(t, 0)}\right) = 0\,.$$

Outline

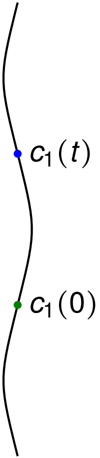
Action of a kinetic gas

Action for a single point particle:

$$S = m\int_0^t (F \circ c_1)(\tau)\, d\tau\,.$$

Assume arc length parameter τ:

$$S = mt\,.$$



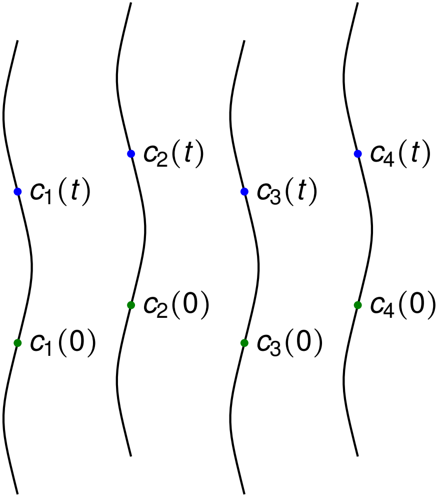
Action of a kinetic gas

Action for P point particles:

$$S_{\text{gas}} = m\sum_{i=1}^{P}\int_0^t (F \circ c_i)(\tau)\,d\tau\,.$$

Assume arc length parameter τ:

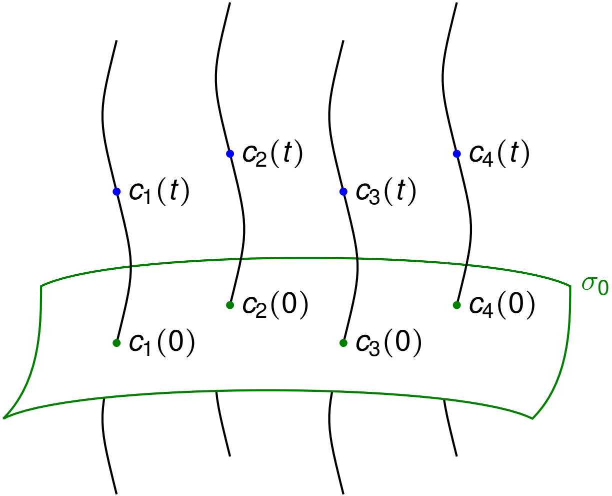
$$S_{\text{gas}} = Pmt\,.$$



Action of a kinetic gas

- Hypersurface of starting points:

$$c_i(0) \in \sigma_0 \,.$$



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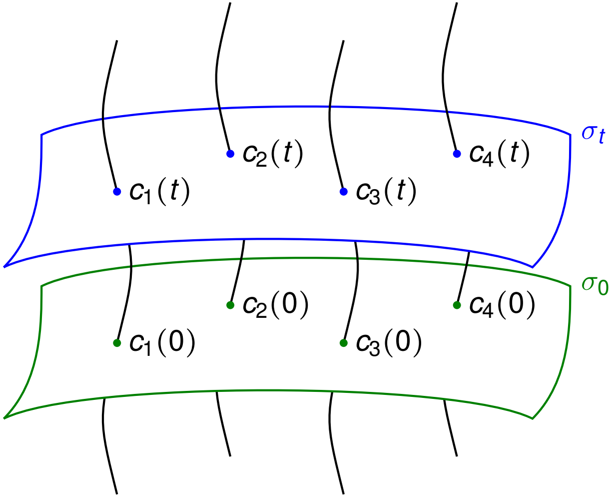

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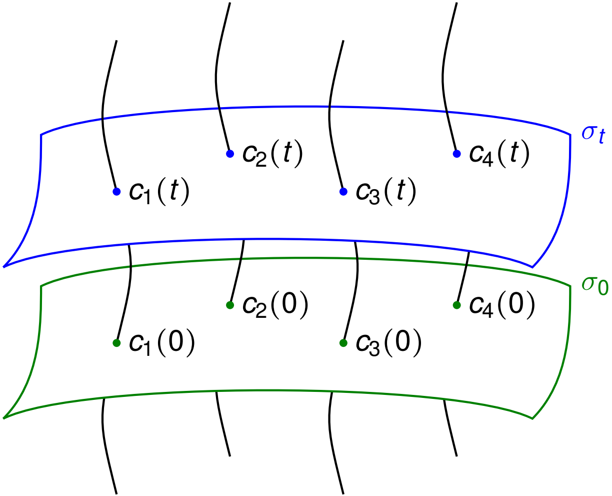
$$P = N[\sigma_\tau] = \int_{\sigma_\tau} \phi\Omega\,.$$



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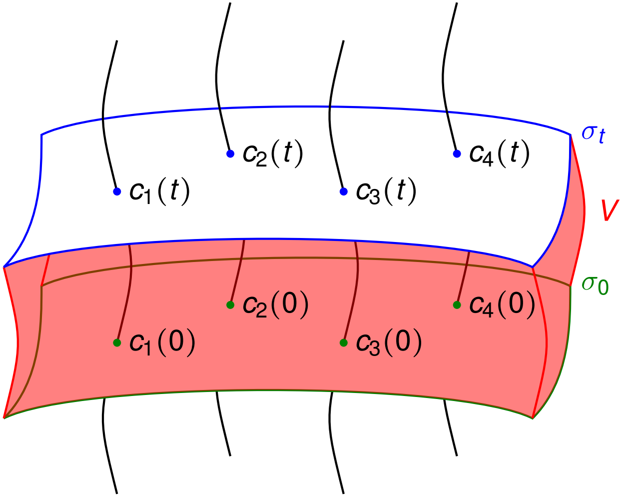

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$$S_{\text{gas}} = Pmt = m\int_0^t \left(\int_{\sigma_\tau} \phi\Omega\right) d\tau$$
$$= m\int_V \phi\Omega \wedge \omega$$
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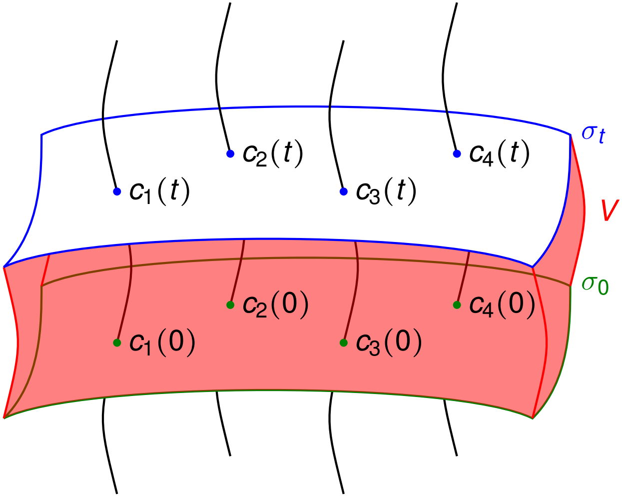
Defined through 1-PDF ϕ.



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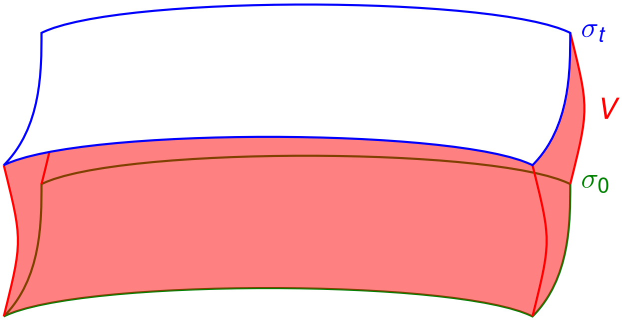
$\Rightarrow$ Forget particle trajectories!



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⇒ Reduces to Einstein-Hilbert action for metric geometry.

Variation and field equations

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$\Rightarrow$ Gravitational field equations with kinetic gas matter:

$$\frac{1}{2} g^{F\,ab} \bar{\partial}_a \bar{\partial}_b (F^2 R_0) - 3R_0 - g^{F\,ab} (\nabla_{\delta_a} P_b - P_a P_b + \bar{\partial}_a (\nabla P_b)) = -\kappa^2 \phi$$

Physical implications

- There are no metric non-vacuum solutions to the field equations.
  - Field equations in case of a metric geometry $F^2 = g_{ab}(x)y^a y^b$:

$$3r_{ab}(x)y^a y^b - r(x)g_{ab}(x)y^a y^b = -\kappa^2\phi g_{ab}(x)y^a y^b\,.$$

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  ↯ Unphysical velocity distribution: uniform over all (arbitrarily high) velocities!

$\Rightarrow$ Gravitational field of a kinetic gas always depends on the velocity of the observer.
  - For observers whose velocity exceeds that of any gas particles:

$$\frac{1}{2}g^{F\,ab}\bar{\partial}_a\bar{\partial}_b(F^2 R_0) - 3R_0 - g^{F\,ab}(\nabla_{\delta_a}P_b - P_a P_b + \bar{\partial}_a(\nabla P_b)) \to 0$$

  - Solution of the differential equation still depends on ϕ via boundary conditions.

  $\Rightarrow$ Observers at velocities beyond gas velocities are still affected, but differently.

Outline

Conclusion

- Summary:
  - Finsler spacetimes:
    - ⋆ Define geometry by length functional.
    - ⋆ Observer space O of physical four-velocities (future unit timelike vectors).
    - ⋆ Geodesics are integral curves of Reeb vector field $\mathbf{r}$ on O.

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- Outlook:
  - Cosmological solutions with non-metric geometry.
    - ⋆ Dark energy?
    - ⋆ Inflation?
  - Extension of parameterized post-Newtonian formalism.

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One-sentence summary

How to summarize this talk in one sentence?

Finsler gravity and the kinetic gas are the most natural description for a gravitating many-particle system.

Special thanks to the following women in science:

- Emmy Noether - for the study of symmetries and conserved quantities in Lagrangian systems and the constructive method to find them.
- Solange F. Rutz - for proposing a Finsler gravity equation, which gave rise to the Finsler gravity action by using the method of variational completion.
- Nicoleta Voicu - for developing the method of variational completion of differential equations, a proper definition of Finsler spacetime, and bringing these ideas together.