Kinetic theory of fluids in Finsler geometry

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Center of Excellence “The Dark Side of the Universe”

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Outline

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Problems in gravity and cosmology

- So far unexplained cosmological observations:
  - Accelerating expansion of the universe
  - Homogeneity of cosmic microwave background

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- Idea here: modification of the geometrical structure of spacetime!
  - Replace metric spacetime geometry by Finsler geometry.
  - Similarly: replacing flat spacetime by curved spacetime led to GR.

Fluids are everywhere

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  - No shear stress, no friction.
  - Characterized by density ρ and pressure p.
    - Dust, dark matter: $p = 0$.
    - Radiation: $p = \frac{1}{3}\rho$.
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- Maxwell-Boltzmann gas:
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- Charged, multi-component gas:
  - Plasma, interacting gas including recombination / ionization.
  - Used in stellar dynamics, pre-CMB era models...

Finsler spacetimes

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  - Other matter field theories
- Possible explanations of yet unexplained phenomena:
  - Fly-by anomaly
  - Galaxy rotation curves
  - Accelerating expansion of the universe
  - Inflation

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The clock postulate

- Proper time along a curve in Lorentzian spacetime:

$$\tau = \int_{t_1}^{t_2} \sqrt{-g_{ab}(x(t))\dot{x}^a(t)\dot{x}^b(t)}dt\,.$$

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- Finsler geometry: use a more general length functional:

$$\tau = \int_{t_1}^{t_2} F(x(t), \dot{x}(t))dt\,.$$

- Finsler function $F : TM \to \mathbb{R}^+$.
- Parametrization invariance requires homogeneity:

$$F(x, \lambda y) = \lambda F(x, y) \quad \forall \lambda > 0\,.$$

Finsler spacetime geometry

- Finsler geometries suitable for spacetimes exist. [C. Pfeifer, M. Wohlfarth '11]

⇒ Finsler metric with Lorentz signature:

$$g^F_{ab}(x,y) = \frac{1}{2}\frac{\partial}{\partial y^a}\frac{\partial}{\partial y^b}F^2(x,y)\,.$$

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- Unit vectors $y \in T_xM$ defined by

$$F^2(x,y) = g^F_{ab}(x,y)y^ay^b = 1\,.$$

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- Ω_x contains a closed connected component $S_x \subseteq \Omega_x$.

⇝ Causality: S_x corresponds to physical observers.

Geometry on the tangent bundle

- Cartan non-linear connection:

$$N^a{}_b = \frac{1}{4}\bar{\partial}_b\left[g^{F\,ac}(y^d\partial_d\bar{\partial}_c F^2 - \partial_c F^2)\right]$$

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⇒ Split of the tangent and cotangent bundles:

- Tangent bundle: $TTM = HTM \oplus VTM$

$$\delta_a = \partial_a - N^b{}_a\bar{\partial}_b\,, \quad \bar{\partial}_a$$

- Cotangent bundle: $T^*TM = H^*TM \oplus V^*TM$

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- Geodesic spray:

$$\mathbf{S} = y^a \delta_a$$

Geometry on observer space

- Recall from the definition of Finsler spacetimes:
  - Set $\Omega_x \subset T_xM$ of unit timelike vectors at $x \in M$.
  - Physical observers correspond to $S_x \subseteq \Omega_x$.
- Definition of observer space:

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- Geodesic hypersurface measure $\omega = \iota_{\mathbf{r}}\Sigma$.
- Note that $\mathcal{L}_{\mathbf{r}}\Sigma = 0$ and $d\omega = 0$.

From metric to Finsler geometry

- Tangent bundle geometry:
  - Finsler function:
  $$F(x,y) = \sqrt{|g_{ab}(x)y^a y^b|}$$
  - Finsler metric:
  $$g^F_{ab}(x,y) = \begin{cases} -g_{ab}(x) & y \text{ timelike} \\ g_{ab}(x) & y \text{ spacelike} \end{cases}$$
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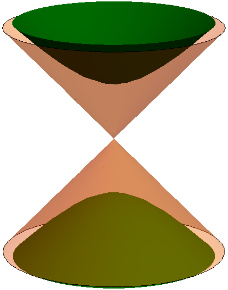
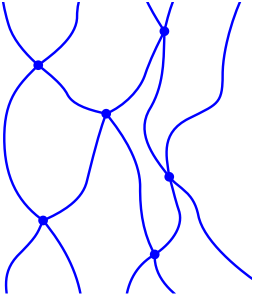


- Observer space:
  - Space Ω_x of unit timelike vectors at $x \in M$.
  - Space S_x of future unit timelike vectors at $x \in M$.
  - Observer space O: union of shells S_x.

Outline

Definition of fluids

- Single-component fluid:
  - Constituted by classical, relativistic particles.
  - Particles have equal properties (mass, charge, ...).
  - Particles follow piecewise geodesic curves.
  - Endpoints of geodesics are interactions with other particles.



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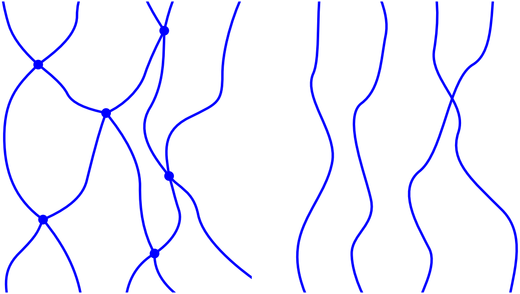
  - Particles do not interact with other particles.
  - $\Rightarrow$ Particles follow geodesics.



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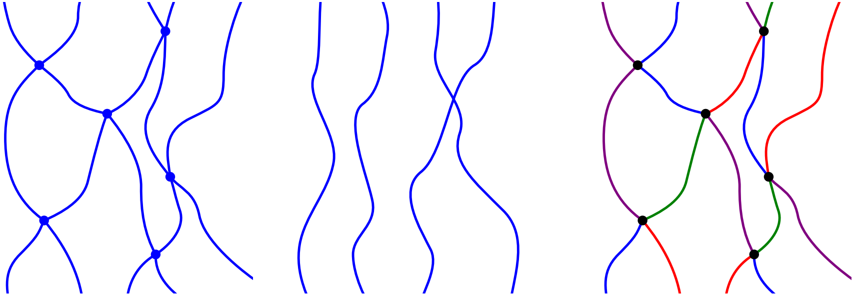
  - ⇒ Particles follow geodesics.
- Multi-component fluid: multiple types of particles.



Geodesics on observer space

- Dynamics of fluids depends on geodesic equation.
- Geodesic equation for curve $x(\tau)$ on spacetime M:

$$\ddot{x}^a + N^a{}_b(x, \dot{x})\dot{x}^b = 0\,.$$

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- Tangent vectors are future unit timelike: $(x, y) \in O$.

⇒ Particle trajectories are piecewise integral curves of $\mathbf{r}$ on O.

One-particle distribution function

- Recall: $\omega = \iota_{\mathbf{r}}\Sigma \in \Omega^6(O)$ unique 6-form such that:
  - ω non-degenerate on every hypersurface not tangent to $\mathbf{r}$.
  - $d\omega = 0$.

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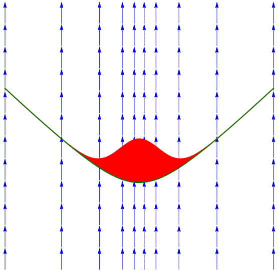
- Define one-particle distribution function $\phi : O \to \mathbb{R}^+$ such that:

For every hypersurface $\sigma \subset O$,

$$N[\sigma] = \int_\sigma \phi\omega$$

# of particle trajectories through σ.



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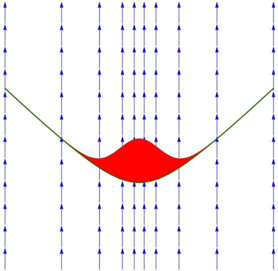

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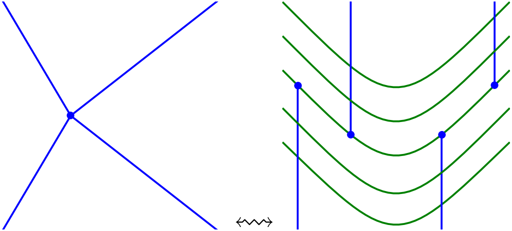
  - Counting of particle trajectories respects hypersurface orientation.
- For multi-component fluids: ϕ_i for each component i.

Collisions & the Liouville equation

- Collision in spacetime $\leftrightsquigarrow$ interruption in observer space.



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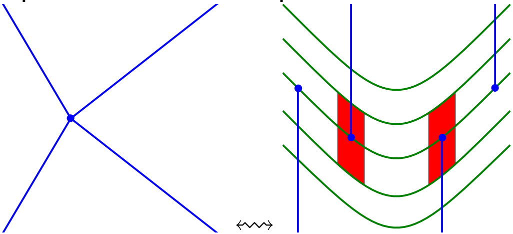

- For any open set $V \in O$,

$$\int_{\partial V} \phi\omega = \int_V d(\phi\omega) = \int_V \mathcal{L}_{\mathbf{r}}\phi\Sigma$$

# of outbound trajectories - # of inbound trajectories.

$\Rightarrow$ Collision density measured by $\mathcal{L}_{\mathbf{r}}\phi$.

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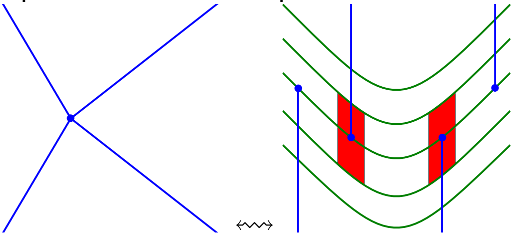

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$\Rightarrow$ Collision density measured by $\mathcal{L}_{\mathbf{r}}\phi$.

- Collisionless fluid: trajectories have no endpoints, $\mathcal{L}_{\mathbf{r}}\phi = 0$.

$\Rightarrow$ Simple, first order equation of motion for collisionless fluid.

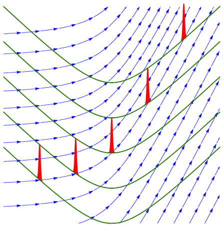
$\Rightarrow$ ϕ is constant along integral curves of $\mathbf{r}$.

Outline

Some (very) pictorial examples

Geodesic dust fluid:

$\phi(x, y) \sim \delta(y - u(x))$.



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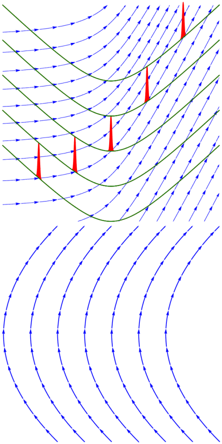

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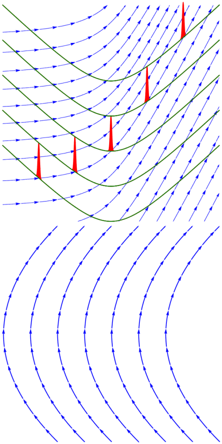

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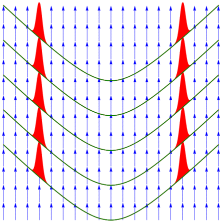

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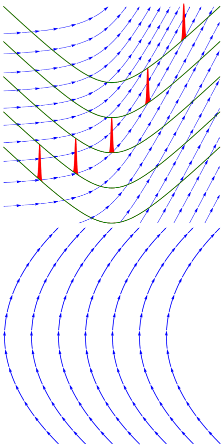

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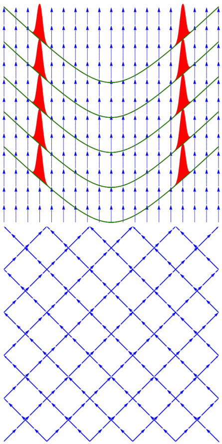

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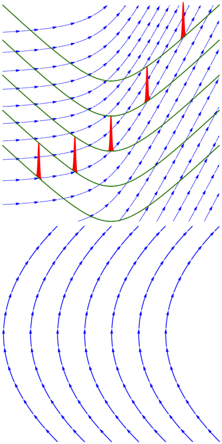

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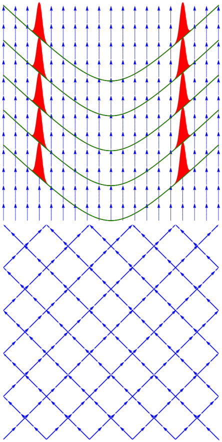

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Interacting fluid:

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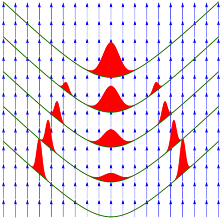

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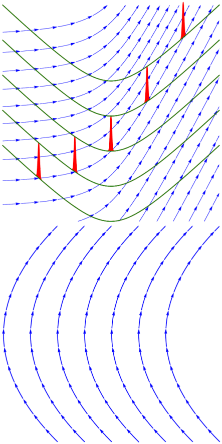

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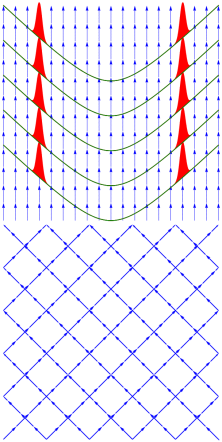

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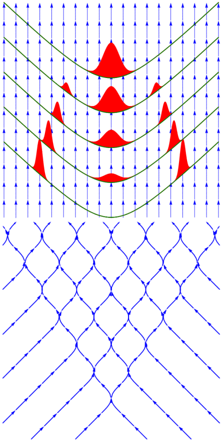

"Humppa"

Example: collisionless dust fluid

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$\Rightarrow$ Generalized (pressureless) Euler equations to Finsler geometry.

- Metric limit $F^2(x,y) = |g_{ab}(x)y^a y^b|$ yields Euler equations:

$$u^b\nabla_b u^a = 0\,, \quad \nabla_a(\rho u^a) = 0\,.$$

Cosmological symmetry

- Introduce suitable coordinates on TM:

$$t, r, \theta, \varphi, y^t, y^r, y^\theta, y^\varphi \,.$$

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- Homogeneity of Finsler function $F(t, y^t, w) = y^t \tilde{F}(t, w/y^t)$.
- Introduce new coordinates: $\tilde{y} = y^t \tilde{F}(t, w/y^t)$, $\tilde{w} = w/y^t$.

$\Rightarrow$ Coordinates on observer space O with $\tilde{y} \equiv 1$.

$\Rightarrow$ Geometry function $\tilde{F}(t, \tilde{w})$ on O.

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- Example: collisionless dust fluid $\phi(x, y) \sim \rho(x)\delta_{S_x}(y, u(x))$:

$$u(t) = \frac{1}{\tilde{F}(t, 0)}\partial_t\,, \quad \partial_t\left(\rho(t)\sqrt{g^F(t, 0)}\right) = 0\,.$$

Outline

Conclusion

- Summary:
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- Outlook:
  - Coupling of fluids to non-metric gravity theories.
  - Cosmological solutions with non-metric geometry.
    - Dark energy?
    - Inflation?
  - Extension of parameterized post-Newtonian formalism.

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