Projective bundle approach to Finsler geometry

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Center of Excellence “The Dark Side of the Universe”

European Union
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Outline

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Motivation

- General relativity:
  - Describes gravitational interaction as geometric phenomenon.
  - Spacetime modeled by smooth manifold M.
  - Geometry described by pseudo-Riemannian metric g.
  - Dynamics defined by Einstein-Hilbert action:

$$S_{\text{EH}}[g] = \int_M \sqrt{-\det g}\, R(g)\, d^4x\,.$$

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  - Generalization of (pseudo-)Riemannian geometry.
  - Yields suitable causal structure for matter field equations.
  - Defines observer clocks via length of their trajectories.
  - Describes point-particle dynamics by Finsler geodesics.

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- Lagrangian formulation of Finsler gravity theories?

Length measure approach to Finsler geometry

- Spacetime geometry defined by function $L : \overset{\circ}{T}M \to \mathbb{R}$:
  - Slit tangent bundle $\overset{\circ}{T}M = TM \setminus \{0_x \in T_xM, x \in M\}$.
  - Homogeneity of degree $h \geq 2$: $L(\lambda v) = \lambda^h L(v)$ for $\lambda \in \mathbb{R}^+$.
  - Zeros of L related to null cones / causal structure.
  - Finsler function $F = |L|^{1/h}$ measures length.

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- Problems:
  - Variations δL must preserve homogeneity of degree h of L.
  - Domain of δL is composed of rays $[v] = \{\lambda v, \lambda \in \mathbb{R}^+\}$ for $v \in \overset{\circ}{T}M$.

  $\Rightarrow$ No variations with compact support.

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⇒ Pullback along section $\sigma : PM \to \overset{\circ}{T}M$ is independent of σ.

⇒ Possible to consider integration on compact subsets of PM.

- Problems:
  - Higher order tensor fields instead of scalar Finsler function.
  - Variation must be constrained for tensor fields to remain Finslerian.

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Positive projective tangent bundle

- Slit tangent bundle $\overset{\circ}{T}M$ carries right action of $\mathbb{R}^+$:

$$\begin{array}{rrcl} \cdot & : \quad \overset{\circ}{T}M \times \mathbb{R}^+ & \to & \overset{\circ}{T}M \\ & (v, \lambda) & \mapsto & v \cdot \lambda = \lambda v \end{array} .$$

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- Consider positive projective tangent bundle PM:

$$PM = \{[v], v \in \mathring{T}M\}, \quad [v] = \{\lambda v, \lambda \in \mathbb{R}^+\} \subset \mathring{T}M .$$

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$\Rightarrow$ Construction defines a principal $\mathbb{R}^+$-bundle with right action $\cdot$:

$$\begin{array}{cccccc} \vartheta & : & \overset{\circ}{T}M & \to & PM \\ & & v & \mapsto & \vartheta(v) = [v] \end{array} .$$

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- Consider left action of $\mathbb{R}^+$ on $\mathbb{R}$:

$$\begin{array}{rccl} \varrho_h & : & \mathbb{R}^+ \times \mathbb{R} & \to & \mathbb{R} \\ & & (\lambda, z) & \mapsto & \varrho_h(\lambda, z) = \lambda^{-h} z \end{array} .$$

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- Homogeneity of L is equivariance with respect to group actions:

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- Finsler geometry models gauge theory of the group $\mathbb{R}^+$.

Associated bundle construction

- Define associated bundle $(Y_h, PM, \pi_h, \mathbb{R})$:
  - Total space $Y_h = \mathring{T}M \times_{\varrho_h} \mathbb{R}$.
  - Base space PM.
  - Bundle map $\pi_h : Y_h \to PM$.
  - Typical fiber $\mathbb{R}$.

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- Elements of the total space are equivalence classes:

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Equivalence of equivariant maps and bundle sections

There exists a one-to-one correspondence between h-homogeneous functions $L : \overset{\circ}{T}M \to \mathbb{R}$ and sections $\hat{L} : PM \to Y_h$:

$$L(v) = z \quad \Leftrightarrow \quad \hat{L}([v]) = [v, z]\,.$$

Variational principle for projective Finsler function

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- Possible to construct Lagrangian also using projective approach?

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- For $\lambda \in \mathbb{R}^+$, consider homothetic transformation:

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- h-homogeneous tensor field $Q : \overset{\circ}{T}M \to T^r_s \overset{\circ}{T}M$: for all $\lambda \in \mathbb{R}^+$

$$(\Phi^{r,s}_\lambda)^{-1} \circ Q \circ \varphi_\lambda = \varphi^*_\lambda Q = \lambda^h Q .$$

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- Relation to Liouville vector field $\mathbf{c} : \overset{\circ}{T}M \to T\overset{\circ}{T}M$:

$$\mathcal{L}_{\mathbf{c}} Q = hQ.$$

Homogeneity and equivariance

- (Left) group action $\rho_h : \mathbb{R}^+ \times T^r_s \mathring{T}M \to T^r_s \mathring{T}M$ such that

$$\varphi_\lambda^* Q = \lambda^h Q \Leftrightarrow \rho_h(\lambda^{-1}, Q(v)) = Q(v \cdot \lambda) = \Phi_\lambda^{r,s}(\lambda^h Q(v)).$$

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$\Rightarrow$ $(Y_h, PM, \pi_h, \mathbb{R}) \cong (Y^{0,0}_h, PM, \pi^{0,0}_h, \mathbb{R}^{(2n)^0})$.

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Pullback bundle vs. fibered product

- Definition of a pullback bundle:
  - Smooth manifolds M, N.
  - Fiber bundle $\pi : E \to M$.
  - Smooth map $\phi : N \to M$.
  - Pullback bundle $\phi^*\pi : \phi^*E \to N$, where
    - total space: $\phi^*E = \{(p, e) \in N \times E, \phi(p) = \pi(e)\}$,
    - projection: $\phi^*\pi(p, e) = p$.
  - Isomorphisms between fibers $F \cong (\phi^*E)_p \cong E_{\phi(p)}$.
  - Fiber bundle structure of E induces fiber bundle structure on ϕ^*E:

$$
\begin{array}{ccc}
\pi^{-1}(U) & \xrightarrow{\psi} & U \times F \\
\downarrow{\scriptstyle \pi} & \swarrow{\scriptstyle \mathrm{pr}_1} & \\
U & &
\end{array}
\quad \Rightarrow \quad
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where U trivializes E around $\phi(p)$ and $\tilde{\psi}(p, e) = (p, \mathrm{pr}_2(\psi(e)))$.

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- For $N = E$ and $\phi = \pi$: $\phi^*E = E \times_M E$.

D-tensors

- Definition of d-tensors:
  - (Slit) tangent bundle: $\overset{(\circ)}{\tau} : \overset{(\circ)}{T}M \to M$.
  - Pullback bundle: $\varpi = \mathring{\tau}^*\tau : \mathring{T}M \times_M TM \to \mathring{T}M$.
  - Tensor bundles: $\mathcal{T}^r_s(\varpi) \cong (\mathring{T}M \times_M TM)^{\otimes r} \otimes (\mathring{T}M \times_M T^*M)^{\otimes s}$.
  - (r, s)-d-tensor field: section of $\mathcal{T}^r_s(\varpi)$.

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  - (r, s)-d-tensor field: section of $\mathcal{T}^r_s(\varpi)$.
- Relation to the double tangent bundle $\psi : T\overset{\circ}{T}M \to \overset{\circ}{T}M$:
  - Canonical injective strong bundle map:

$$\begin{array}{rcccc} \mathbf{i} & : & \overset{\circ}{T}M \times_M TM & \to & T\overset{\circ}{T}M \\ & & (v, w) & \mapsto & \frac{d}{dt}(v + tw)\big|_{t=0} \end{array}$$

  - Canonical surjective strong bundle map:

$$\begin{array}{rcccc} \mathbf{j} & : & T\overset{\circ}{T}M & \to & \overset{\circ}{T}M \times_M TM \\ & & \xi & \mapsto & (\psi(\xi), \overset{\circ}{\tau}_*(\xi)) \end{array}$$

Homogeneous d-tensors

- Maps form exact sequence:

$$0 \to \mathring{T}M \times_M TM \xrightarrow{\mathbf{i}} T\mathring{T}M \xrightarrow{\mathbf{j}} \mathring{T}M \times_M TM \to 0$$

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- Dual exact sequence:

$$0 \leftarrow \overset{\circ}{T}M \times_M T^*M \xleftarrow{\mathbf{i}^*} T^*\overset{\circ}{T}M \xleftarrow{\mathbf{j}^*} \overset{\circ}{T}M \times_M T^*M \leftarrow 0$$

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- Use maps **i** and $\mathbf{j}^*$ to map d-tensors to $T^r_s\overset{\circ}{T}M$.

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- Use maps **i** and $\mathbf{j}^*$ to map d-tensors to $T^r_s\overset{\circ}{T}M$.
- Define homogeneity via the image tensor fields $\in \Gamma(T^r_s\overset{\circ}{T}M)$.

Homogeneous d-tensors

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Non-linear connections

- Non-linear connection: splitting of exact sequence

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- Bundle splitting $T\overset{\circ}{T}M = V\overset{\circ}{T}M \oplus H\overset{\circ}{T}M$:
  - $V\overset{\circ}{T}M = \operatorname{im}\mathbf{i} = \operatorname{im}\mathbf{v} = \ker\mathbf{j} = \ker\mathbf{h}$: canonically defined.
  - $H\overset{\circ}{T}M = \operatorname{im}\mathbf{h} = \operatorname{im}\mathcal{H} = \ker\mathbf{v} = \ker\mathcal{V}$: defined only by connection.

Non-linear connections as tensor fields

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- Compare with other structures:
  - Tangent structure $J : T\mathring{T}M \to T\mathring{T}M$ with $\operatorname{im} J = \ker J = V\mathring{T}M$ is -1-homogeneous.
  - Adjoint structure $\Theta : T\mathring{T}M \to T\mathring{T}M$ with $\operatorname{im} J = \ker J = H\mathring{T}M$ is 1-homogeneous.

Outline

Conclusion

- Summary:
  - Homogeneity = equivariance under group action of $\mathbb{R}^+$.
  - Define orbit space $PM = \overset{\circ}{T}M/\mathbb{R}^+$.
  - Homogeneous functions $\leftrightarrow$ sections of $\pi_h : Y_h \to PM$.
  - Describe Finsler geometry in terms of section $\hat{L} : PM \to Y_h$.
  - $\Rightarrow$ Well-defined domains for action integrals.
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- References:

  - M. Hohmann, C. Pfeifer and N. Voicu,
    "Finsler gravity action from variational completion",
    arXiv:1812.11161 [gr-qc] (to appear in Phys. Rev. D).