Harmonic d-tensors

A tool for calculating symmetric Finsler spacetimes

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DPG Spring Conference - Session MP 8 - 3. March 2016

Motivation

- So far unexplained cosmological observations:
  - Accelerating expansion of the universe
  - Homogeneity of cosmic microwave background

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- Idea here: modification of the geometric structure of spacetime!
  - Replace metric spacetime geometry by Finsler geometry.
  - Similarly: replacing flat spacetime by curved spacetime led to GR.

Finsler spacetimes

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  - Approaches to quantum gravity
  - Electrodynamics in anisotropic media
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  - Finsler function measures length of tangent vectors.
  - Well-defined notions of connections, curvature, parallel transport. . .

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- Finsler spacetimes are suitable backgrounds for:
  - Gravity
  - Electrodynamics
  - Other matter field theories
- Possible explanations of yet unexplained phenomena:
  - Fly-by anomaly
  - Galaxy rotation curves
  - Accelerating expansion of the universe

Basics of Finsler gravity

- Finsler geometric spacetime background:
  - Proper time defined by Finsler length functional:

$$\tau = \int_{t_1}^{t_2} F(x(t), \dot{x}(t)) dt \,.$$

  - Finsler function $F : TM \to \mathbb{R}$.
  - Finsler metric with Lorentz signature:

$$g^F_{ab}(x, y) = \frac{1}{2} \frac{\partial}{\partial y^a} \frac{\partial}{\partial y^b} F^2(x, y) \,.$$

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- Defining objects of Finsler geometry are d-tensors.

D-tensors

- Definition of d-tensors:
  - Tangent bundle: $\tau : TM \to M$.
  - Pullback bundle: $\pi : TM \times_M TM \to TM$.
  - Tensor bundles: $\mathcal{T}^r_s(\pi)$.
  - (r, s)-d-tensor field: section of $\mathcal{T}^r_s(\pi)$.

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- Relation to the double tangent bundle $\varpi : TTM \to TM$:
  - Canonical injective strong bundle map:

$$\begin{array}{rcccl} \mathbf{i} & : & TM \times_M TM & \to & TTM \\ & & (v, w) & \mapsto & \frac{d}{dt}(v + tw)\Big|_{t=0} \end{array}$$

  - Canonical surjective strong bundle map:

$$\begin{array}{rcccl} \mathbf{j} & : & TTM & \to & TM \times_M TM \\ & & \xi & \mapsto & (\varpi(\xi), \tau_*(\xi)) \end{array}$$

  - Exact sequence:

$$0 \to TM \times_M TM \xrightarrow{\mathbf{i}} TTM \xrightarrow{\mathbf{j}} TM \times_M TM \to 0$$

  - Vertical tangent bundle: $VTM = \operatorname{im}\mathbf{i} = \ker\mathbf{j}$.

Diffeomorphisms acting on d-tensors

- Lift of diffeomorphisms to d-tensors:
  - Diffeomorphism $\varphi : M \to M$.

  $\Rightarrow$ Lift to the tangent bundle: $\varphi_* : TM \to TM$.

  $\Rightarrow$ Lift to the pullback bundle: $\varphi_* \times_M \varphi_* : TM \times_M TM \to TM \times_M TM$.

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- Infinitesimal diffeomorphisms:
  - Vector field $X = X^a\partial_a \in \mathfrak{X}(M)$.

  $\Rightarrow$ Complete lift $\hat{X} = X^a\partial_a + y^a\partial_a X^b\bar{\partial}_b \in \mathfrak{X}(TM)$.

  $\Rightarrow$ Action on d-tensor $T \in \mathcal{T}^r_s(\pi)$ in coordinate basis of TM:

$$
\begin{aligned}
(\mathcal{L}_{\hat{X}}T)^{a_1\cdots a_r}{}_{b_s} = {} & X^c\partial_c T^{a_1\cdots a_r}{}_{b_1\cdots b_s} + y^d\partial_d X^c\bar{\partial}_c T^{a_1\cdots a_r}{}_{b_1\cdots b_s} \\
& - \partial_c X^{a_1} T^{ca_2\cdots a_r}{}_{b_1\cdots b_s} - \ldots - \partial_c X^{a_r} T^{a_1\cdots a_{r-1}c}{}_{b_1\cdots b_s} \\
& + \partial_{b_1} X^c T^{a_1\cdots a_r}{}_{cb_2\cdots b_s} + \ldots + \partial_{b_s} X^c T^{a_1\cdots a_r}{}_{b_1\cdots b_{s-1}c}
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- Introduce short notation:
  - Vector field $\mathbf{X} \in \mathfrak{X}(TM)$.
  - Corresponding Lie derivative $\mathcal{X} = i\mathcal{L}_{\mathbf{X}}$.

Generating vector fields of SO(3)

- Coordinates on *TM* for $M = \mathbb{R}^3$:

$$r, \bar{\rho}, \bar{z}, \beta, \theta, \phi$$

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$$\begin{aligned}
\mathbf{R}_1 &= \sin\phi\partial_\theta + \frac{\cos\phi}{\tan\theta}\partial_\phi - \frac{\cos\phi}{\sin\theta}\partial_\beta\,,\\
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- Auxiliary vector fields:

$$\mathbf{B}_1 = \sin\beta\partial_\theta + \frac{\cos\beta}{\tan\theta}\partial_\beta - \frac{\cos\beta}{\sin\theta}\partial_\phi\,,$$
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Symmetry algebra for SO(3)

- Rotation algebra:

$$[\mathcal{R}_i, \mathcal{R}_j] = i\epsilon_{ijk}\mathcal{R}_k\,, \quad [\mathcal{B}_i, \mathcal{B}_j] = i\epsilon_{ijk}\mathcal{B}_k\,, \quad [\mathcal{B}_i, \mathcal{R}_j] = 0\,.$$

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- Introduce ladder operators and Casimir:

$$\mathcal{R}_\pm = \mathcal{R}_1 \pm i\mathcal{R}_2\,, \quad \mathcal{R}_z = \mathcal{R}_3\,, \quad \mathcal{B}_\pm = \mathcal{B}_1 \pm i\mathcal{B}_2\,, \quad \mathcal{B}_z = \mathcal{B}_3\,,$$
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⇒ Algebra relations:

$$[\mathcal{R}_z, \mathcal{R}_\pm] = \pm\mathcal{R}_\pm\,, \quad [\mathcal{R}_+, \mathcal{R}_-] = 2\mathcal{R}_z\,, \quad [\mathcal{R}_\pm, \mathcal{R}^2] = [\mathcal{R}_z, \mathcal{R}^2] = 0\,,$$
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⇒ $\mathcal{R}^2, \mathcal{R}_z, \mathcal{B}_z$ mutually commute.

Scalar spherical harmonics on *TM*

- Definition of spherical scalar harmonics:

$$\mathcal{Y}_{l,m,n}(\theta,\phi,\beta) = N_{l,m,n} e^{im\phi} e^{in\beta} \cos^{m+n}\frac{\theta}{2} \sin^{|m-n|}\frac{\theta}{2}$$
$$\cdot {}_2F_1\left(\max(m,n) - l, \max(m,n) + l + 1; |m-n| + 1; \sin^2\frac{\theta}{2}\right)$$

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- Operator relations:
  - Eigenvalue relations:

$$\mathcal{R}^2\mathcal{Y}_{l,m,n} = l(l+1)\mathcal{Y}_{l,m,n}\,, \quad \mathcal{R}_z\mathcal{Y}_{l,m,n} = m\mathcal{Y}_{l,m,n}\,, \quad \mathcal{B}_z\mathcal{Y}_{l,m,n} = n\mathcal{Y}_{l,m,n}$$

  - Ladder operators:

$$\mathcal{R}_\pm\mathcal{Y}_{l,m,n} = \sqrt{(l\mp m)(l\pm m+1)}\mathcal{Y}_{l,m\pm1,n}\,,$$
$$\mathcal{B}_\pm\mathcal{Y}_{l,m,n} = \sqrt{(l\mp n)(l\pm n+1)}\mathcal{Y}_{l,m,n\pm1}\,.$$

Spherical harmonic d-tensors

- Basis $\mathbf{e}_{-1}, \mathbf{e}_0, \mathbf{e}_1$ of $\mathcal{T}^0_1(\pi)$ such that

$$\mathcal{R}^2\mathbf{e}_m = 2\mathbf{e}_m\,, \quad \mathcal{R}_z\mathbf{e}_m = m\mathbf{e}_m\,, \quad \mathcal{R}_\pm\mathbf{e}_m = \sqrt{(1 \mp m)(2 \pm m)}\mathbf{e}_{m\pm 1}$$

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- Zeroth order tensors in $\mathcal{T}^0_0(\pi)$:

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- Zeroth order tensors in $\mathcal{T}^0_0(\pi)$:

$$\overset{m}{\underset{n}{\mathbf{Y}}}{}_l = \mathcal{Y}_{l,m,n}$$

- Recursive definition in $\mathcal{T}^0_k(\pi)$:

$$\overset{m}{\underset{n}{\mathbf{Y}}}{}_{l_0 l_1 \cdots l_k} = (-1)^{l_k - m}\sqrt{2l_k + 1}\sum_{m',\mu}\begin{pmatrix} l_k & l_{k-1} & 1 \\ m & -m' & -\mu \end{pmatrix}\overset{m'}{\underset{n}{\mathbf{Y}}}{}_{l_0 l_1 \cdots l_{k-1}} \otimes \mathbf{e}_\mu$$

Spherical harmonic d-tensors

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- Analogue construction for dual basis and mixed tensors.

Spherical operator relations

- Eigenvalue relations:

$$\mathcal{R}^2 \overset{m}{\underset{n}{\mathbf{Y}}}{}_{l_0 l_1 \cdots l_k} = l_k(l_k+1) \overset{m}{\underset{n}{\mathbf{Y}}}{}_{l_0 l_1 \cdots l_k}\,,$$

$$\mathcal{R}_z \overset{m}{\underset{n}{\mathbf{Y}}}{}_{l_0 l_1 \cdots l_k} = m \overset{m}{\underset{n}{\mathbf{Y}}}{}_{l_0 l_1 \cdots l_k}\,.$$

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Application example: Finsler metric

- Vertical gradient operator for $f = f(r, \bar{\rho}, \bar{z})$:

$$\nabla^v \left(f \overset{m}{\underset{n}{\mathbf{Y}}}_{l_0 l_1 \cdots l_k} \right) =$$

$$\left[\frac{1}{\sqrt{2}} \left(n \frac{f}{\bar{\rho}} - f_{\bar{\rho}} \right) \overset{0}{\underset{1}{\mathbf{Y}}}{}_1{}^0 + \frac{1}{\sqrt{2}} \left(n \frac{f}{\bar{\rho}} + f_{\bar{\rho}} \right) \overset{0}{\underset{-1}{\mathbf{Y}}}{}_1{}^0 - f_{\bar{z}} \overset{0}{\underset{0}{\mathbf{Y}}}{}_1{}^0 \right] \otimes \overset{m}{\underset{n}{\mathbf{Y}}}_{l_0 l_1 \cdots l_k} .$$

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- Finsler metric for $L(r, \bar{\rho}, \bar{z}) = F^2(r, \bar{\rho}, \bar{z})$:

$$\begin{aligned} g^F &= \frac{1}{2} \nabla^v \nabla^v L \\ &= -\frac{1}{2\sqrt{3}} \left(\frac{L_{\bar{\rho}}}{\bar{\rho}} + L_{\bar{\rho}\bar{\rho}} + L_{\bar{z}\bar{z}} \right) \overset{0}{\underset{0}{\mathbf{Y}}}{}_0{}^{1\,0} - \frac{1}{2\sqrt{6}} \left(\frac{L_{\bar{\rho}}}{\bar{\rho}} + L_{\bar{\rho}\bar{\rho}} - 2L_{\bar{z}\bar{z}} \right) \overset{0}{\underset{0}{\mathbf{Y}}}{}_2{}^{1\,0} \\ &\quad + \frac{1}{2} L_{\bar{\rho}\bar{z}} \left(\overset{0}{\underset{1}{\mathbf{Y}}}{}_2{}^{1\,0} - \overset{0}{\underset{-1}{\mathbf{Y}}}{}_2{}^{1\,0} \right) + \frac{1}{4} \left(L_{\bar{\rho}\bar{\rho}} - \frac{L_{\bar{\rho}}}{\bar{\rho}} \right) \left(\overset{0}{\underset{2}{\mathbf{Y}}}{}_2{}^{1\,0} + \overset{0}{\underset{-2}{\mathbf{Y}}}{}_2{}^{1\,0} \right) . \end{aligned}$$

Generating vector fields of SO(4)

- Coordinates on TM for $M = \mathbb{R}^4$:

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$$\begin{aligned}
\mathbf{J}_1^\pm &= \sin\phi^\pm \partial_{\theta^\pm} + \frac{\cos\phi^\pm}{\tan\theta^\pm}\partial_{\phi^\pm} - \frac{\cos\phi^\pm}{\sin\theta^\pm}\partial_\beta\,,\\
\mathbf{J}_2^\pm &= -\cos\phi^\pm \partial_{\theta^\pm} + \frac{\sin\phi^\pm}{\tan\theta^\pm}\partial_{\phi^\pm} - \frac{\sin\phi^\pm}{\sin\theta^\pm}\partial_\beta\,,\\
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- Auxiliary vector field:

$$\mathbf{B} = -\partial_\beta\,.$$

Symmetry algebra for SO(4)

- Rotation algebra:

$$[\mathcal{J}_i^\pm, \mathcal{J}_j^\pm] = i\epsilon_{ijk}\mathcal{J}_k^\pm\,, \quad [\mathcal{J}_i^+, \mathcal{J}_j^-] = 0\,, \quad [\mathcal{B}, \mathcal{J}_i^\pm] = 0\,.$$

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- Introduce ladder operators and Casimirs:

$$\mathcal{J}_{\pm}^{\pm} = \mathcal{J}_1^{\pm} \pm i\mathcal{J}_2^{\pm}\,, \quad \mathcal{J}_z^{\pm} = \mathcal{J}_3^{\pm}\,,$$
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⇒ Algebra relations:

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⇒ $(\mathcal{J}^\pm)^2, \mathcal{J}_z^\pm, \mathcal{B}$ mutually commute.

Scalar cosmological harmonics on *TM*

- Definition of cosmological scalar harmonics:

$$\mathcal{Z}_{l^+,l^-,m^+,m^-,n}(\theta^+,\theta^-,\phi^+,\phi^-,\beta) = N_{l^+,l^-,m^+,m^-,n} e^{im^+\phi^+} e^{im^-\phi^-} e^{in\beta}$$
$$\cdot {}_2F_1\left(\max(m^+,n) - l, \max(m^+,n) + l + 1; |m^+ - n| + 1; \sin^2\frac{\theta^+}{2}\right)$$
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- Operator relations:
  - Eigenvalue relations:

$$(\mathcal{J}^{\pm})^2\mathcal{Z} = l^{\pm}(l^{\pm}+1)\mathcal{Z}\,, \quad \mathcal{J}_z^{\pm}\mathcal{Z} = m^{\pm}\mathcal{Z}\,, \quad \mathcal{B}\mathcal{Z} = n\mathcal{Z}$$

  - Ladder operators:

$$\mathcal{J}_{\pm}^{+}\mathcal{Z}_{l^+,l^-,m^+,m^-,n} = \sqrt{(l^+ \mp m^+)(l^+ \pm m^+ + 1)}\,\mathcal{Z}_{l^+,l^-,m^+\pm1,m^-,n}\,,$$
$$\mathcal{J}_{\pm}^{-}\mathcal{Z}_{l^+,l^-,m^+,m^-,n} = \sqrt{(l^- \mp m^-)(l^- \pm m^- + 1)}\,\mathcal{Z}_{l^+,l^-,m^+,m^-\pm1,n}\,.$$

Cosmological d-tensor basis

- Introduce basis of $\mathcal{T}^0_1$:

$$\mathbf{e}_{\frac{1}{2},\frac{1}{2}}\,,\quad \mathbf{e}_{-\frac{1}{2},\frac{1}{2}}\,,\quad \mathbf{e}_{\frac{1}{2},-\frac{1}{2}}\,,\quad \mathbf{e}_{-\frac{1}{2},-\frac{1}{2}}$$

- Operator relations:

$$(\mathcal{J}^{\pm})^2\mathbf{e}_{m^+,m^-} = \frac{3}{4}\mathbf{e}_{m^+,m^-}\,,\quad \mathcal{J}^{\pm}_z\mathbf{e}_{m^+,m^-} = m^{\pm}\mathbf{e}_{m^+,m^-}\,,$$

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- Introduce basis of $\mathcal{T}^1_0$:

$$\mathbf{e}^{\frac{1}{2},\frac{1}{2}}\,, \quad \mathbf{e}^{-\frac{1}{2},\frac{1}{2}}\,, \quad \mathbf{e}^{\frac{1}{2},-\frac{1}{2}}\,, \quad \mathbf{e}^{-\frac{1}{2},-\frac{1}{2}}$$

- Operator relations:

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- Analogue construction for dual basis.

Recursive construction of cosmological d-tensors

- Zeroth order tensors in $\mathcal{T}_0^0(\pi)$:

$$\overset{m^+,m^-}{\underset{n}{\mathbf{Z}}}\{l^+,l^-\} = \mathcal{Z}_{l^+,l^-,m^+,m^-,n}$$

- Recursive definition in $\mathcal{T}_k^0(\pi)$:

$$\overset{m^+,m^-}{\underset{n}{\mathbf{Z}}}\{l_0^+,l_0^-\}_{\{l_1^+,l_1^-\}\cdots\{l_k^+,l_k^-\}} = (-1)^{l_k^+ + l_k^- - m^+ - m^-}\sqrt{2l_k^+ + 1}\sqrt{2l_k^- + 1}$$

$$\cdot \sum_{m^{+\prime},m^{-\prime},\mu^+,\mu^-} \begin{pmatrix} l_k^+ & l_{k-1}^+, & \frac{1}{2} \\ m^+ & -m^{+\prime} & -\mu^+ \end{pmatrix} \begin{pmatrix} l_k^- & l_{k-1}^-, & \frac{1}{2} \\ m^- & -m^{-\prime} & -\mu^- \end{pmatrix}$$

$$\cdot \overset{m^{+\prime},m^{-\prime}}{\underset{n}{\mathbf{Z}}}\{l_0^+,l_0^-\}_{\{l_1^+,l_1^-\}\cdots\{l_{k-1}^+,l_{k-1}^-\}} \otimes \mathbf{e}_{\mu^+,\mu^-}\,,$$

Cosmological operator relations

- Eigenvalue relations:

$$(\mathcal{J}^{\pm})^2 \overset{m^+,m^-}{\underset{n}{\mathbf{Z}}}{}_{\{l_0^+,l_0^-\}\{l_1^+,l_1^-\}\cdots\{l_k^+,l_k^-\}} = l_k^{\pm}(l_k^{\pm}+1) \overset{m^+,m^-}{\underset{n}{\mathbf{Z}}}{}_{\{l_0^+,l_0^-\}\{l_1^+,l_1^-\}\cdots\{l_k^+,l_k^-\}}\,,$$

$$\mathcal{J}_z^{\pm} \overset{m^+,m^-}{\underset{n}{\mathbf{Z}}}{}_{\{l_0^+,l_0^-\}\{l_1^+,l_1^-\}\cdots\{l_k^+,l_k^-\}} = m^{\pm} \overset{m^+,m^-}{\underset{n}{\mathbf{Z}}}{}_{\{l_0^+,l_0^-\}\{l_1^+,l_1^-\}\cdots\{l_k^+,l_k^-\}}\,.$$

- Ladder operators:

$$\mathcal{J}_{\pm}^{+} \overset{m^+,m^-}{\underset{n}{\mathbf{Z}}}{}_{\{l_0^+,l_0^-\}\{l_1^+,l_1^-\}\cdots\{l_k^+,l_k^-\}} = \sqrt{(l_k^+ \mp m^+)(l_k^+ \pm m^+ + 1)} \overset{m^+\pm 1,m^-}{\underset{n}{\mathbf{Z}}}{}_{\{l_0^+,l_0^-\}\{l_1^+,l_1^-\}\cdots\{l_k^+,l_k^-\}}\,,$$

$$\mathcal{J}_{\pm}^{-} \overset{m^+,m^-}{\underset{n}{\mathbf{Z}}}{}_{\{l_0^+,l_0^-\}\{l_1^+,l_1^-\}\cdots\{l_k^+,l_k^-\}} = \sqrt{(l_0^- \mp m^-)(l_0^- \pm m^- + 1)} \overset{m^+,m^-\pm 1}{\underset{n}{\mathbf{Z}}}{}_{\{l_0^+,l_0^-\}\{l_1^+,l_1^-\}\cdots\{l_k^+,l_k^-\}}\,.$$

Summary

- Finsler geometry in physics:
  - Finsler geometry well established in electrodynamics, astronomy...
  - Finsler spacetimes generalizes pseudo-Riemannian spacetimes.
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- Harmonic d-tensors:
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  - Simple calculation rules for operators in Finsler geometry.
  - Simplify calculation of d-tensors in Finsler gravity.

Outlook

- Construction of harmonic d-tensors:
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- Applications:
  - Finsler gravity - cosmology, spherically symmetric solutions…
  - Finsler electrodynamics.
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Acknowledgments

- Estonian Research Council
  - ERMOS115 “Geometric extensions of general relativity - foundations and phenomenology”
  - PUT790 “Geometric foundations of gravity and their comparison to observations”
- Archimedes Foundation
  - TK133 “The Dark Side of the Universe”

Job openings supported by these grants:

- PhD student position (deadline: 20. March 2016)
- PostDoc position (deadline: 10. April 2016)

http://www.fi.ut.ee/en/postdoc-and-phd-in-gravity-theory