Fluid dynamics on Finsler spacetimes and cosmology

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  - Modification of the laws of gravity?
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- Idea here: modification of the geometrical structure of spacetime!
  - Replace metric spacetime geometry by Finsler geometry.
  - Similarly: replacing flat spacetime by curved spacetime led to GR.

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  - Other matter field theories
- Possible explanations of yet unexplained phenomena:
  - Fly-by anomaly
  - Galaxy rotation curves
  - Accelerating expansion of the universe
  - Inflation

The clock postulate

- Proper time along a curve in Lorentzian spacetime:

$$\tau = \int_{t_1}^{t_2} \sqrt{-g_{ab}(x(t))\dot{x}^a(t)\dot{x}^b(t)}dt\,.$$

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- Finsler geometry: use a more general length functional:

$$\tau = \int_{t_1}^{t_2} F(x(t), \dot{x}(t))dt\,.$$

- Finsler function $F : TM \to \mathbb{R}^+$.
- Parametrization invariance requires homogeneity:

$$F(x, \lambda y) = \lambda F(x, y) \quad \forall \lambda > 0\,.$$

Finsler spacetimes

- Finsler geometries suitable for spacetimes exist. [C. Pfeifer, M. Wohlfarth '11]

⇒ Finsler metric with Lorentz signature:

$$g^F_{ab}(x,y) = \frac{1}{2}\frac{\partial}{\partial y^a}\frac{\partial}{\partial y^b}F^2(x,y)\,.$$

⇒ Notion of timelike, lightlike, spacelike tangent vectors.

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- Unit vectors $y \in T_xM$ defined by

$$F^2(x,y) = g^F_{ab}(x,y)y^a y^b = 1\,.$$

⇒ Set $\Omega_x \subset T_xM$ of unit timelike vectors at $x \in M$.

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- Ω_x contains a closed connected component $S_x \subseteq \Omega_x$.

$\rightsquigarrow$ Causality: S_x corresponds to physical observers.

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$$\left.\left. + 2g^{F\,ab}\left(\nabla_a S_b + S_a S_b + \bar{\partial}_a(y^c\delta_c S_b - N^c{}_b S_c)\right)\right]\right|_\Sigma = \kappa T|_\Sigma$$

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- Geometry side obtained by variation of S_G with respect to F.
- Variation of matter action yields energy-momentum scalar T.

Point masses on Finsler spacetimes

- Point masses follow Finsler geodesics.
- Geodesic equation for curve $x(\tau)$ on spacetime M:

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$$x\,, \quad y = \dot{x} \in O = \bigcup_{x \in M} S_x \subset TM\,.$$

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$\Rightarrow$ Solutions are integral curves of vector field on O:

$$y^a\partial_a - y^b N^a{}_b\bar{\partial}_a = \mathbf{r}\,.$$

$\Rightarrow$ Point mass trajectories modeled by integral curves of $\mathbf{r}$ on O.

Fluids on Finsler spacetimes

- Single-component fluid:
    - Constituted by classical, relativistic particles.
    - Particles have equal properties (mass, charge, . . .).
    - Particles follow piecewise geodesic curves.
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- Continuum limit:
  - Phase space O is filled with particles.
  - Particle density function $\phi : O \to \mathbb{R}^+$.
- Collisionless fluid:
  - Particles do not interact with other particles.

  $\Rightarrow$ Particles follow geodesics.

  $\Rightarrow$ Continuum dynamics given by Liouville equation:

$$\mathcal{L}_{\mathbf{r}}\phi = 0\,.$$

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- Metric limit $F^2(x, y) = |g_{ab}(x) y^a y^b|$ yields Euler equations:

$$u^b \nabla_b u^a = 0\,, \quad \nabla_a(\rho u^a) = 0\,.$$

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  - Density ρ.
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$$T_{\rho,p,u}(x,y) = (1-6(g_{ab}(x)u^a(x)y^b)^2)\rho(x)+3(1-2(g_{ab}(x)u^a(x)y^b)^2)p(x)\,.$$

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- Generalize to Finsler fluid:
  - Consider dust: $p = 0$.
  - Consider superposition of dust with different velocities.
  - Integrate over contributions from each velocity.
  - Generalize $g_{ab}u^a v^b$ to Finsler angle.

$$T_\phi(x,v) = m\int_{S_x} d^3v'\sqrt{\det h(x,v')}\phi(x,v')(1-6\cos^2\sphericalangle(v,v'))\,.$$

Cosmological symmetry

- Introduce suitable coordinates on *TM*:

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- Most general Finsler function obeying cosmological symmetry:

$$F = F(t, y^t, w)\,, \quad w^2 = \frac{(y^r)^2}{1 - kr^2} + r^2\left((y^\theta)^2 + \sin^2\theta(y^\varphi)^2\right)\,.$$

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- Homogeneity of Finsler function $F(t, y^t, w) = y^t \tilde{F}(t, w/y^t)$.
- Introduce new coordinates: $\tilde{y} = y^t \tilde{F}(t, w/y^t)$, $\tilde{w} = w/y^t$.

$\Rightarrow$ Coordinates on observer space O with $\tilde{y} \equiv 1$.

$\Rightarrow$ Geometry function $\tilde{F}(t, \tilde{w})$ on O.

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- Example: collisionless dust fluid $\phi(x, y) \sim \rho(x)\delta_{S_x}(y, u(x))$:

$$u(t) = \frac{1}{\tilde{F}(t, 0)}\partial_t , \quad \partial_t\left(\rho(t)\sqrt{g^F(t, 0)}\right) = 0 .$$

Cosmological gravitational dynamics

- Start from gravitational field equations:

$$\left[g^{F\,ab}\bar{\partial}_a\bar{\partial}_b(R^c{}_{cd}y^d) - 6\frac{R^a{}_{ab}y^b}{F^2} + 2g^{F\,ab}\left(\nabla_a S_b + S_a S_b + \bar{\partial}_a(y^c\delta_c S_b - N^c{}_b S_c)\right)\right]\Bigg|_\Sigma = \kappa T|_\Sigma$$

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- Some terms simplify for cosmological symmetry: $R^a{}_{ab}y^b$.
- Some terms don't simplify at all: $N^a{}_b$, $\nabla_a S_b$.
- Simplify the problem:
  - Finsler perturbation of metric geometry.
  - Finsler function using higher rank tensors: $H_{a_1\cdots a_n}y^{a_1}\cdots y^{a_n}$.

Summary

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  - Define geometry by length functional.
  - Observer space O of physical four-velocities.
  - Geodesics are integral curves of vector field on O.
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  - Define fluid density on observer space.
  - Collisionless fluid satisfies Liouville equation.
- Cosmology:
  - All quantities depend on only two coordinates $t, \tilde{w}$.
  - Simple equation of motion for cosmological fluid matter.
  - Gravitational field equation becomes involved.

Outlook

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  - Simple model Finsler geometry from higher rank tensors.
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  - Finsler perturbation of metric background.
  - Simple model Finsler geometry from higher rank tensors.
  - Fully general Finsler-Friedmann equations?
- Solving for cosmological dynamics
  - Dark energy?
  - Inflation?