Fluid dynamics on Finsler spacetimes

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Motivation

- Fluids are everywhere:
  - Perfect fluid (radiation, dust, dark matter...) - cosmology.
  - Maxwell-Boltzmann gas - atmospheres.
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  - All measurements are performed by observers.
  - Measurements depend on observer's frame (velocity).
  - Quantum gravity: possible non-tensorial velocity dependence.
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- Finsler spacetimes as observer space geometry:
  - Finsler geometry of space widely used in physics.
  - Finsler geometry generalizes Riemannian geometry.
  - Finsler spacetimes are suitable backgrounds for physics.
  - Possible explanations of yet unexplained phenomena.

Finsler spacetime geometry

- Generalize metric length measure to Finsler function:

$$\tau = \int_{t_1}^{t_2} \sqrt{|g_{ab}(x(t))\dot{x}^a(t)\dot{x}^b(t)|}dt \rightsquigarrow \int_{t_1}^{t_2} F(x(t), \dot{x}(t))dt\,.$$

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$\Rightarrow$ Observer space $O \subset TM$ of allowed velocities.

Point mass dynamics on observer space

- Point mass follows curve $\gamma : \mathbb{R} \to M$ on spacetime M.
- γ is extremal curve of Finsler length measure:

$$\delta \int F(\gamma(t), \dot{\gamma}(t)) dt = 0 \, .$$

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- Canonical lift Γ of curve to tangent bundle TM:

$$\Gamma = (\gamma, \dot{\gamma})\,.$$

- Lift of geodesic equation to TM is first order differential equation:

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- Physically allowed velocities: $\Gamma(t) \in O$.
- Restriction $\mathbf{r} = \mathbf{S}|_O$: Reeb vector field.

$\Rightarrow$ Physical geodesics are integral curves of $\mathbf{r}$ on O.

From particles to fluids

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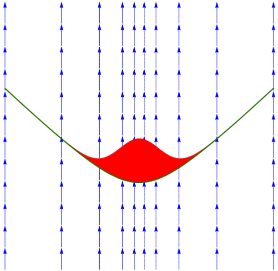
- Define one-particle distribution function $\phi : O \to \mathbb{R}^+$ such that:



For every hypersurface $\sigma \subset O$,

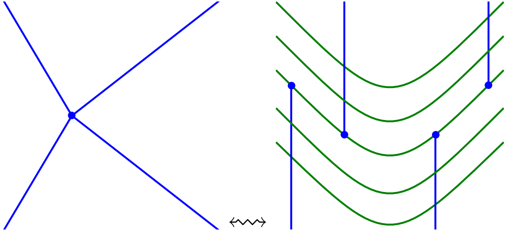
$$N[\sigma] = \int_\sigma \phi\omega$$

# of particle trajectories through σ.

  -
  - Counting of particle trajectories respects hypersurface orientation.

Collisions & the Liouville equation

- Collision in spacetime $\leftrightsquigarrow$ interruption in observer space.



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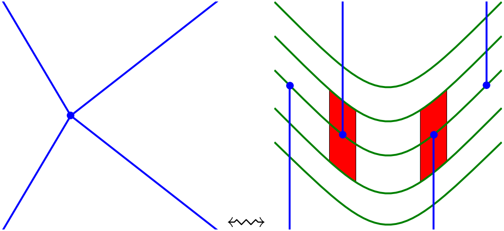

- For any open set $V \in O$,

$$\int_{\partial V} \phi\omega = \int_V d(\phi\omega) = \int_V \mathcal{L}_{\mathbf{r}}\phi\Sigma$$

# of outbound trajectories - # of inbound trajectories.

$\Rightarrow$ Collision density measured by $\mathcal{L}_{\mathbf{r}}\phi$.

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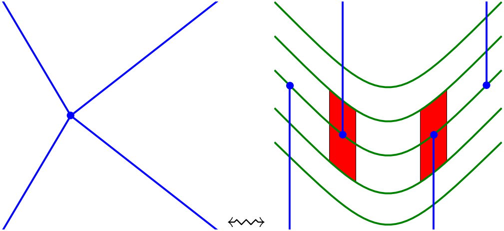

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- Collisionless fluid: trajectories have no endpoints, $\mathcal{L}_{\mathbf{r}}\phi = 0$.

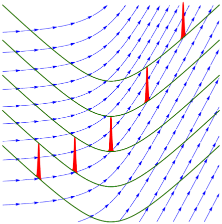
$\Rightarrow$ Simple, first order equation of motion for collisionless fluid.

$\Rightarrow$ ϕ is constant along integral curves of $\mathbf{r}$.

Examples of fluids

Geodesic dust fluid:

$\phi(x,y) \sim \delta(y-u(x))\,.$



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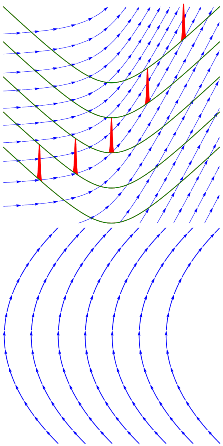

“Jenkka”

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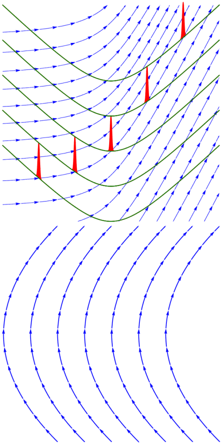

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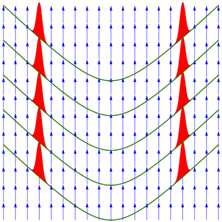

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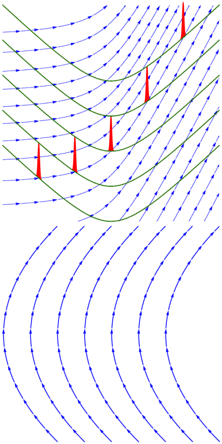

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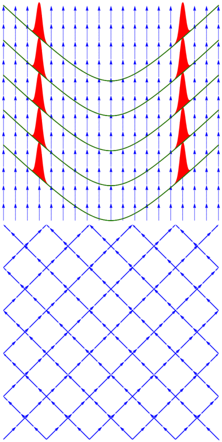

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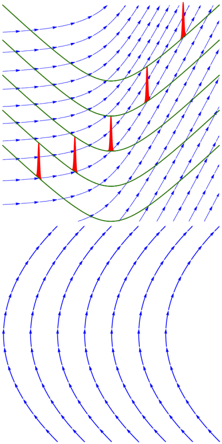

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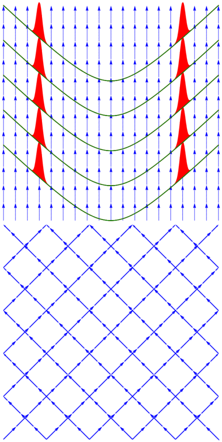

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Interacting fluid:

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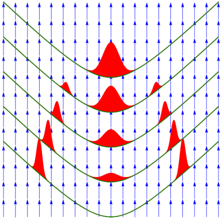

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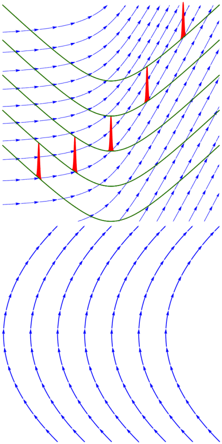

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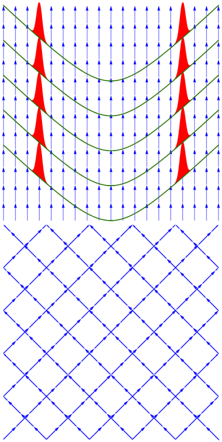

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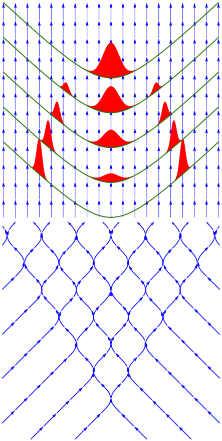

"Humppa"

Example: dust fluid in Finsler geometry

- Classical dust with density $\rho(x)$ and velocity $u^a(x)$.
- One-partical distribution function on O:

$$\phi(\hat{x}, \theta) = \frac{1}{m}\rho(\hat{x})\frac{\delta(\theta - v(\hat{x}))}{\sqrt{\det h^F(\hat{x}, \theta)}}\,.$$

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- No collisions $\Rightarrow$ Liouville equation $\mathcal{L}_{\mathbf{r}}\phi = 0$.

$\Rightarrow$ Equations of motion for ρ and u^a:

$$\nabla u^a = 0 \quad \text{and} \quad \nabla_{\delta_a}(\rho u^a) = 0\,.$$

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  - Horizontal part of Cartan linear connection ∇_{δ_a}.
- Metric background geometry $F(x,y) = \sqrt{|g_{ab}(x)y^a y^b|}$:

$$u^b\nabla_b u^a = 0 \quad \text{and} \quad \nabla_a(\rho u^a) = 0\,.$$

$\Rightarrow$ Well-known Euler equations of fluid dynamics.

Fluids with cosmological symmetry

- Most general cosmological Finsler function $F(\hat{t}, \hat{y}, \hat{w})$.
  - Cosmological time $\hat{t}$.
  - Velocity component $\hat{y}$ in $\hat{t}$-direction.
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- Robertson-Walker metric: $\tilde{F} = \sqrt{1 - a^2(\hat{t})\hat{w}^2/\hat{y}^2}$:

$$\phi_t = -\frac{\hat{w}}{\hat{y}}\left(\frac{\hat{w}^2}{\hat{y}^2}a^2 - 2\right)\frac{\dot{a}}{a}\phi_w\,.$$

Conclusion

- Basic idea:
  - Model fluids by particle trajectories.
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  - Describe geometry of observer space using Finsler geometry.
  - Measure particle density by distribution function.
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- Future research goals:
  - Coupling of fluids to non-metric gravity theories.
  - Cosmological solutions of gravity with non-metric geometry.
  - Extension of parameterized post-Newtonian formalism.

References

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