Extensions of Lorentzian spacetime geometry

From Finsler to Cartan and vice versa

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15. November 2013

Outline

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References

- This work:
  - MH,
    "Extensions of Lorentzian spacetime geometry:
    from Finsler to Cartan and vice versa,"
    Phys. Rev. D **87** (2013) 124034 [arXiv:1304.5430 [gr-qc]].
- Cartan geometry of observer space:
  - S. Gielen and D. K. Wise,
    "Lifting General Relativity to Observer Space,"
    J. Math. Phys. **54** (2013) 052501 [arXiv:1210.0019 [gr-qc]].
- Finsler spacetimes (see preceding talk by C. Pfeifer):
  - C. Pfeifer and M. N. R. Wohlfarth,
    "Causal structure and electrodynamics on Finsler spacetimes,"
    Phys. Rev. D **84** (2011) 044039 [arXiv:1104.1079 [gr-qc]].
  - C. Pfeifer and M. N. R. Wohlfarth,
    "Finsler geometric extension of Einstein gravity,"
    Phys. Rev. D **85** (2012) 064009 [arXiv:1112.5641 [gr-qc]].

Physical motivation

- A simple experiment: particle propagation in spacetime (M, g).
  - A supernova occurs at some “beacon” event $x_0 \in M$.
  - Neutrinos from the supernova follow geodesics γ in M.
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$\Rightarrow$ Consider observer frames as more fundamental than spacetime.

$\Rightarrow$ Spacetime emerges from equivalence classes of observer frames.

- Geometric theory based on this assumption?

Why Cartan geometry on observer space?

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  - Loop quantum gravity
  - Spin foam models
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- Solution:
  - Consider space O of all allowed observers.
  - Describe experiments on observer space instead of spacetime.

  $\Rightarrow$ Observer dependence of physical quantities follows naturally.

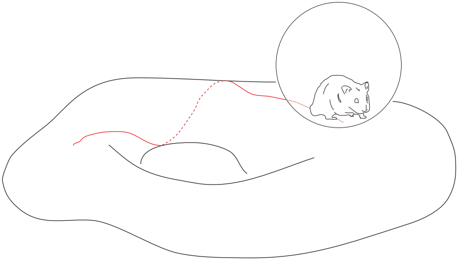
  $\Rightarrow$ No preferred observers.
  - Geometry of observer space modeled by Cartan geometry.

Why Finsler geometry of spacetimes?

- See previous talk by C. Pfeifer.
- Finsler geometry of space widely used in physics:
  - Approaches to quantum gravity
  - Electrodynamics in anisotropic media
  - Modeling of astronomical data
- Finsler geometry generalizes Riemannian geometry:
  - Clock postulate: proper time equals arc length along trajectories.
  - Geometry described by Finsler metric.
  - Well-defined notions of connections, curvature, parallel transport…
- Finsler spacetimes are suitable backgrounds for:
  - Gravity
  - Electrodynamics
  - Other matter field theories
- Possible explanations of yet unexplained phenomena:
  - Fly-by anomaly
  - Galaxy rotation curves
  - Accelerating expansion of the universe

Outline

Toy model: The hamster ball



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  - Orthonormal frame bundle $\pi : P \to M$ is principal SO(2)-bundle.
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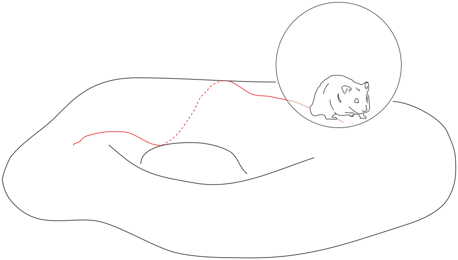

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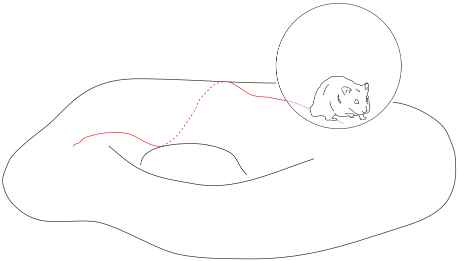

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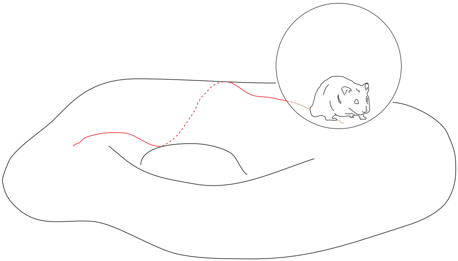

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$\Rightarrow$ Surface M "traced" by $S^2 \cong \mathrm{SO}(3)/\mathrm{SO}(2) = G/H$.

$\Rightarrow$ Geometry of M fully determines Hamster ball motion.

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Brief summary

- Finsler geometry defined by length functional:

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- Set $\Omega_x \subset T_xM$ of unit timelike vectors at $x \in M$.
- Ω_x contains a closed connected component $S_x \subseteq \Omega_x$.

$\rightsquigarrow$ Causality: S_x corresponds to physical observers.

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Observer space

- Recall from the definition of Finsler spacetimes:
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- Tangent vectors $y \in S_x$ satisfy $g^F_{ab}(x, y)y^a y^b = 1$.

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- Let P be the space of all observer frames.

$\Rightarrow$ $\pi : P \to O$ is a principal SO(3)-bundle.

- In general no principal $\mathrm{SO}_0(3,1)$-bundle $\tilde{\pi} : P \to M$.

Cartan connection

- Need to construct $A \in \Omega^1(P, \mathfrak{g})$.
- Recall that

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- Coefficients of Cartan linear connection:

$$N^a{}_b = \frac{1}{4} \bar{\partial}_b \left[g^{F\,aq} \left(y^p \partial_p \bar{\partial}_q F^2 - \partial_q F^2 \right) \right] \,,$$

$$F^a{}_{bc} = \frac{1}{2} g^{F\,ap} \left(\delta_b g^F_{pc} + \delta_c g^F_{bp} - \delta_p g^F_{bc} \right) \,,$$

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Fundamental vector fields

- Let $a = z^i \mathcal{Z}_i + \frac{1}{2} h^i{}_j \mathcal{H}_i{}^j \in \mathfrak{g}$.
- Define the vector field

$$\underline{A}(a) = z^i f_i^a \left(\partial_a - f_j^b F^c{}_{ab} \bar{\partial}_c^j \right) + \left(h^i{}_j f_i^a - h^i{}_0 f_i^b f_j^c C^a{}_{bc} \right) \bar{\partial}_a^j .$$

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⇒ For all $p \in P$ we find

$$A(\underline{A}(a)(p)) = a\,.$$

⇒ For all $w \in T_pP$ we find

$$\underline{A}(A(w))(p) = w\,.$$

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- Vertical vector fields $\underline{A}(\mathfrak{h})$: Lorentz transforms.

Time translation

- Consider the fundamental vector field of the time translator $\mathcal{Z}_0$,
$$\mathbf{t} = \underline{A}(\mathcal{Z}_0) = f_0^a \partial_a - f_j^a N^b{}_a \bar{\partial}_b^j \qquad \Leftrightarrow \qquad \omega^i{}_j(\mathbf{t}) = 0\,, \quad e^i(\mathbf{t}) = \delta_0^i\,.$$
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- From $\omega^i{}_0(\mathbf{t}) = 0$ follows:

$$0 = \dot{f}_0^a + N^a{}_b \dot{x}^b = \ddot{x}^a + N^a{}_b \dot{x}^b\,.$$

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$\Rightarrow$ Integral curves of $\mathbf{t}$ define freely falling observers.

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- $R^d{}_{cab}$, $P^d{}_{cab}$, $S^d{}_{cab}$: curvature of Cartan linear connection.

$\Rightarrow$ Cartan geometry reproduces well-known Finsler objects.

Outline

Spacetime

- Start from observer space Cartan geometry $(\pi : P \to O, A)$.
- ↯ *Condition 1*: boost distribution $\underline{A}(\mathfrak{h})$ must be integrable.

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↯ *Condition 2*: foliation $\mathcal{F}$ must be strictly simple.

$\Rightarrow$ Leaf space M is a smooth manifold.

$\Rightarrow$ Canonical projection $\tilde{\pi} : P \to M$ is a submersion.

- Canonical projections $\tilde{\pi} = \pi' \circ \pi$:

$$P \xrightarrow{\pi} O \xrightarrow{\pi'} M, \qquad P \xrightarrow{\tilde{\pi}} M$$

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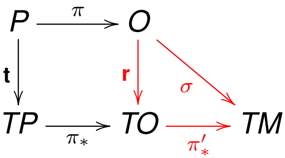

- Integral curves $\lambda \mapsto o(\lambda) \in O$ of $\mathbf{r}$ must be canonical lifts:

$$\sigma(o(\lambda)) = \frac{d}{d\lambda}\pi'(o(\lambda)) = \pi'_*(\dot{o}(\lambda)) = \pi'_*(\mathbf{r}(o(\lambda)))\,.$$

$\Rightarrow$ Uniquely defined map $\sigma = \pi'_* \circ \mathbf{r}$.

$\lightning$ *Condition 3*: σ must be an embedding.

Finsler geometry

- Finsler function must be positively homogeneous of degree one:

$$F(x, \lambda y) = |\lambda| F(x, y)$$

- Unit timelike condition: $F(\sigma(o)) = 1$ for all observers $o \in O$.

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$\Rightarrow$ Finsler spacetime geometry on $\mathbb{R}\sigma(O)$.

- No Finsler geometry on $TM \setminus \mathbb{R}\sigma(O)$.
- Cartan geometry describes only geometry visible to observers.

Outline

Reconstruction of a given Finsler spacetime

- Idea:
  - Start from a Finsler spacetime (M, F).
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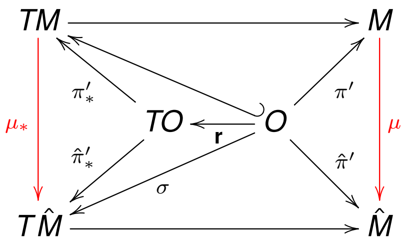

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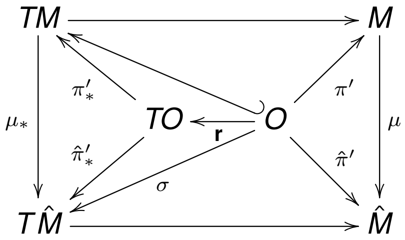

- μ preserves the Finsler function on timelike vectors.

$\Rightarrow$ Reconstruction of the original Finsler geometry.

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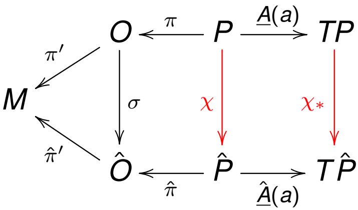

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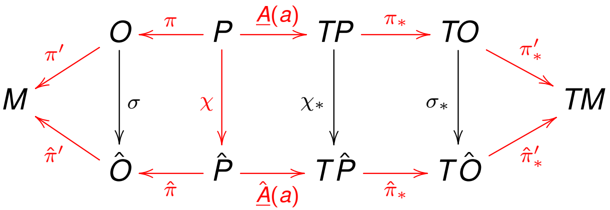

- Every Cartan morphism $\chi = (x, f)$ takes the form

$$x(p) = \pi'(\pi(p)), \quad f_i(p) = \pi'_*(\pi_*(\underline{A}(\mathcal{Z}_i)(p)))$$

$\Rightarrow$ Simple test for equivalence of $(\pi : P \to O, A)$ and $(\hat{\pi} : \hat{P} \to \hat{O}, \hat{A})$.

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Gravity from Cartan to Finsler

- MacDowell-Mansouri gravity on observer space: [S. Gielen, D. Wise '12]

$$S_G = \int_O \epsilon_{\alpha\beta\gamma} \, \mathrm{tr}_{\mathfrak{h}}(F_{\mathfrak{h}} \wedge \star F_{\mathfrak{h}}) \wedge b^\alpha \wedge b^\beta \wedge b^\gamma$$

  - Hodge operator $\star$ on $\mathfrak{h}$.
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- Translate terms into Finsler language (with $R = d\omega + \frac{1}{2}[\omega, \omega]$):
  - Curvature scalar:

$$[e, e] \wedge \star R \rightsquigarrow g^{F\,ab} R^c{}_{acb}\, dV\,.$$

  - Cosmological constant:

$$[e, e] \wedge \star [e, e] \rightsquigarrow dV\,.$$

  - Gauss-Bonnet term:

$$R \wedge \star R \rightsquigarrow \epsilon^{abcd} \epsilon^{efgh} R_{abef} R_{cdgh}\, dV\,.$$

$\Rightarrow$ Gravity theory on Finsler spacetime.

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  - Geometrodynamics of Finsler spacetimes.
  - ...