Extensions of Lorentzian spacetime geometry: From Finsler to Cartan and vice versa

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Motivation

- Quantum gravity may break symmetries:
  - General covariance.
  - Local Lorentz invariance.
- Observer-dependent physical quantities.
- Consider space of physical observers.
- Geometry of observer space?

Observer space Cartan geometry

- G/H Cartan geometry:
  - Lie groups G and $H \subset G$.
  - Principal H-bundle $\pi : P \to M$.
  - Cartan connection $A \in \Omega^1(P, \mathfrak{g})$.
- Choose $K = \mathrm{SO}(3)$, $H = \mathrm{SO}_0(3,1)$, $G \in \{\mathrm{SO}_0(4,1), \mathrm{ISO}_0(3,1), \mathrm{SO}_0(3,2)\}$.
- Lorentzian manifold (M, g).
- Future unit timelike vectors $O \subset TM$.
- Space P of orthonormal frames.
- Principal K-bundle $\pi : P \to O$.
- G/K Cartan geometry on $\pi : P \to O$ [1].

Finsler spacetimes

- Finsler length functional:

$$\tau = \int_{t_1}^{t_2} F(x(t), \dot{x}(t))dt\,.$$

- Finsler function $F : TM \to \mathbb{R}^+$.
- Geometry suitable for spacetimes [2, 3].
- Finsler metric with Lorentz signature:

$$g^F_{ab}(x,y) = \frac{1}{2}\frac{\partial}{\partial y^a}\frac{\partial}{\partial y^b}F^2(x,y)\,.$$

- Unit timelike vectors $\Omega_x \subset T_xM$.
- Closed connected component $S_x \subseteq \Omega_x$.
- S_x corresponds to physical observers.

Summary

- Finsler geometry $\Leftrightarrow$ Cartan geometry.
- Reconstruction of original geometries.
- Translation of gravitational actions.

Outlook

- Derive gravitational equations of motion.
- Consistent matter coupling.
- Study of exact solutions.
- Effects of Finsler metric geometry?
- Finsler-Cartan geometrodynamics?

References

[1] S. Gielen and D. K. Wise, J. Math. Phys. **54** (2013) 052501 [arXiv:1210.0019 [gr-qc]].

[2] C. Pfeifer and M. N. R. Wohlfarth, Phys. Rev. D **84** (2011) 044039 [arXiv:1104.1079 [gr-qc]].

[3] C. Pfeifer and M. N. R. Wohlfarth, Phys. Rev. D **85** (2012) 064009 [arXiv:1112.5641 [gr-qc]].

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From Finsler to Cartan

- Observer space $O \subset TM$:

$$O = \bigcup_{x \in M} S_x\,.$$

- Space P of orthonormal frames f_i:

$$g^F_{ab}(x, f_0)f^a_i f^b_j = -\eta_{ij}\,.$$

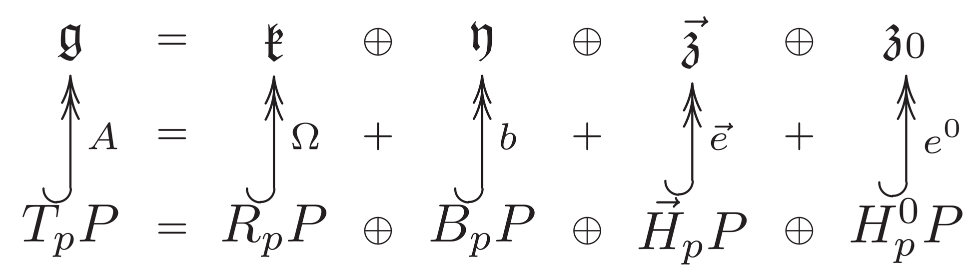
- Principal K-bundle $\pi : P \to O$.
- Cartan connection $A \in \Omega^1(P, \mathfrak{g})$:

$$A = f^{-1}{}^i{}_a dx^a \mathcal{Z}_i + \frac{1}{2}f^{-1}{}^i{}_a\Big[f^b_j\Big(dx^c F^a{}_{bc} + (dx^d N^c{}_d + df^c_0)C^a{}_{bc}\Big) + df^a_j\Big]\mathcal{H}_i{}^j\,.$$

- Fundamental vector fields:

$$\underline{A}(a) = z^i f^a_i\left(\partial_a - f^b_j F^c{}_{ab}\bar{\partial}^j_c\right) + \left(h^i{}_j f^a_i - h^i{}_0 f^b_i f^c_j C^a{}_{bc}\right)\bar{\partial}^j_a$$

  for $a = z^i\mathcal{Z}_i + \frac{1}{2}h^i{}_j\mathcal{H}_i{}^j \in \mathfrak{g}$.
- Split of the tangent bundle TP:



- Time translation:
  - Integral curve (x, f) of $\mathbf{t} = \underline{A}(\mathcal{Z}_0)$.
  - (x, f_0) is canonical lift:

$$e^i(\mathbf{t}) = \delta^i_0 \quad\Rightarrow\quad \dot{x}^a = f^a_0\,.$$

  - (x, f_0) is Finsler geodesic:

$$\omega^i{}_0(\mathbf{t}) = 0 \quad\Rightarrow\quad 0 = \ddot{x}^a + N^a{}_b\dot{x}^b\,.$$

  - Frames are parallely transported:

$$\omega^\alpha{}_\beta(\mathbf{t}) = 0 \quad\Rightarrow\quad 0 = \nabla_{(\dot{x},\dot{f}_0)}f^a_\alpha\,.$$

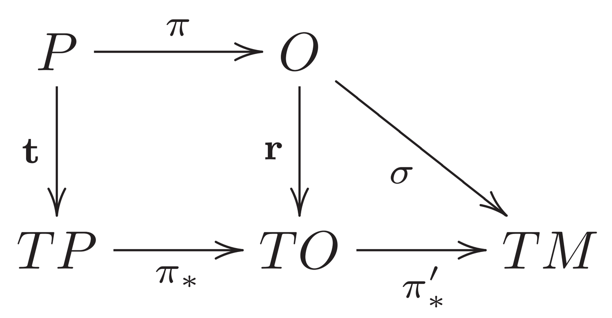
- Curvature $F = dA + \frac{1}{2}[A,A] \in \Omega^2(P, \mathfrak{g})$:

$$F = -f^{-1}{}^i{}_a C^a{}_{bc}\mathcal{Z}_i dx^b \wedge \delta f^c_0 - \frac{1}{4}f^{-1}{}^i{}_d f^c_j \mathcal{H}_i{}^j\Big(R^d{}_{cab}dx^a \wedge dx^b + 2P^d{}_{cab}dx^a \wedge \delta f^b_0 + S^d{}_{cab}\delta f^a_0 \wedge \delta f^b_0\Big)$$

$\Rightarrow$ Cartan geometry from Finsler geometry.

From Cartan to Finsler

- Vertical distribution $VP = RP \oplus BP$.
- VP must be integrable.
- Integrate VP to foliation $\mathcal{F}$.
- $\mathcal{F}$ must be strictly simple.
- Smooth leaf space M of $\mathcal{F}$.
- Embedding $\sigma : O \to TM$ of observers:



- Finsler function F on $\mathbb{R}\sigma(O)$:

$$F(\lambda\sigma(o)) = |\lambda|\,.$$

$\Rightarrow$ Finsler geometry from Cartan geometry.

Finsler $\to$ Cartan $\to$ Finsler

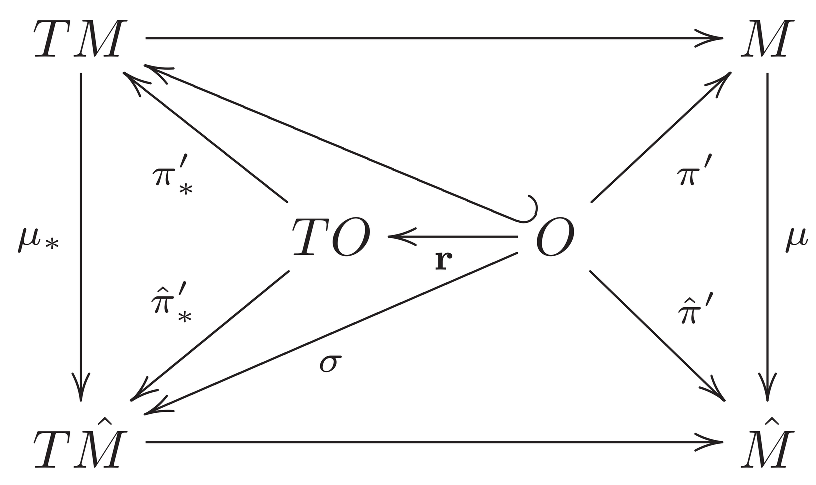
- Finsler spacetime (M, F).

$\to$ Cartan geometry $(\pi : P \to O, A)$.

$\to$ New Finsler spacetime $(\hat{M}, \hat{F})$.

- (M, F) equals $(\hat{M}, \hat{F})$?

$\to$ Finsler diffeomorphism μ *always* exists:



$\Rightarrow$ Reconstruction of Finsler spacetime.

Cartan $\to$ Finsler $\to$ Cartan

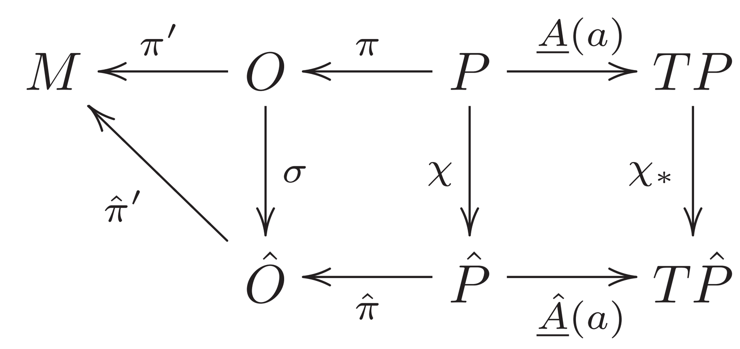
- Cartan geometry $(\pi : P \to O, A)$.

$\to$ Finsler spacetime (M, F).

$\to$ New Cartan geometry $(\hat{\pi} : \hat{P} \to \hat{O}, \hat{A})$.

- $(\pi : P \to O, A)$ equals $(\hat{\pi} : \hat{P} \to \hat{O}, \hat{A})$?

$\to$ *Only if* a "Cartan morphism" χ exists:



- Cartan morphism $\chi = (x, f)$ satisfies

$$x(p) = \pi'(\pi(p))\,,\ \ f_i(p) = \pi'_*(\pi_*(\underline{A}(\mathcal{Z}_i)(p)))$$

$\Rightarrow$ Simple test for equivalence.

MacDowell-Mansouri gravity

- Gravity action on observer space [1]:

$$S_G = \int_O \epsilon_{\alpha\beta\gamma}\,\mathrm{tr}_{\mathfrak{h}}(F_{\mathfrak{h}} \wedge \star F_{\mathfrak{h}}) \wedge b^\alpha \wedge b^\beta \wedge b^\gamma\,.$$

- Translate terms to Finsler language:
  - Curvature scalar:

$$[e,e] \wedge \star R \rightsquigarrow g^{F\,ab}R^c{}_{acb}\,dV\,.$$

  - Cosmological constant:

$$[e,e] \wedge \star[e,e] \rightsquigarrow dV\,.$$

  - Gauss-Bonnet term:

$$R \wedge \star R \rightsquigarrow \epsilon^{abcd}\epsilon^{efgh}R_{abef}R_{cdgh}\,dV\,.$$

$\Rightarrow$ Gravity theory on Finsler spacetime.

Finsler gravity

- Finsler gravity action [3]:

$$S_G = \int_O d^4x\,d^3y\,\sqrt{-\tilde{G}}R^a{}_{ab}y^b\,.$$

- Translate terms to Cartan language:

$$d^4x\,d^3y\,\sqrt{-\tilde{G}} = \epsilon_{ijkl}\epsilon_{\alpha\beta\gamma}\,b^\alpha \wedge b^\beta \wedge b^\gamma \wedge e^i \wedge e^j \wedge e^k \wedge e^l\,,$$

$$R^a{}_{ab}y^b = b^\alpha[\underline{A}(\mathcal{Z}_\alpha), \underline{A}(\mathcal{Z}_0)]\,.$$

$\Rightarrow$ Gravity theory on observer space.