Extensions of Lorentzian spacetime geometry

From Finsler to Cartan and vice versa

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Outline

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References

- This work:
  - MH,
    "Extensions of Lorentzian spacetime geometry:
    from Finsler to Cartan and vice versa,"
    arXiv:1304.5430 [gr-qc].
- Cartan geometry of observer space:
  - S. Gielen and D. K. Wise,
    "Lifting General Relativity to Observer Space,"
    arXiv:1210.0019 [gr-qc].
- Finsler spacetimes:
  - C. Pfeifer and M. N. R. Wohlfarth,
    "Causal structure and electrodynamics on Finsler spacetimes,"
    Phys. Rev. D **84** (2011) 044039 [arXiv:1104.1079 [gr-qc]].
  - C. Pfeifer and M. N. R. Wohlfarth,
    "Finsler geometric extension of Einstein gravity,"
    Phys. Rev. D **85** (2012) 064009 [arXiv:1112.5641 [gr-qc]].

Physical motivation

- A simple experiment: light propagation in spacetime M.
  - A supernova occurs at some “beacon” event $x_0 \in M$.
  - Light from the supernova follows a null geodesic γ in M.
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$\Rightarrow$ Consider observer frames as more fundamental than spacetime.

- Geometric theory based on this assumption?

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- Problems:
  - Breaking of Copernican principle.
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- Solution:
  - Consider space O of all allowed observers.
  - Describe experiments on observer space instead of spacetime.
  - ⇒ Observer dependence of physical quantities follows naturally.
  - ⇒ No preferred observers.
  - Geometry of observer space modeled by Cartan geometry.

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- Finsler geometry of space widely used in physics:
    - Approaches to quantum gravity
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- Possible explanations of yet unexplained phenomena:
  - Fly-by anomaly
  - Galaxy rotation curves
  - Accelerating expansion of the universe

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Cartan geometry

- Ingredients of a Cartan geometry:
  - A Lie group G with a closed subgroup $H \subset G$.
  - A principal H-bundle $\pi : P \to M$.
  - A 1-form $A \in \Omega^1(P, \mathfrak{g})$ on P with values in $\mathfrak{g}$.

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  - For each $p \in P$, $A_p : T_pP \to \mathfrak{g}$ is a linear isomorphism.
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- Fundamental vector fields:
  - $\Rightarrow$ A has an "inverse" $\underline{A} : \mathfrak{g} \to \Gamma(TP)$.
  - $\Rightarrow$ Vector fields $\underline{A}(a)$ for $a \in \mathfrak{g}$ are nowhere vanishing.

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- Curvature of the Cartan connection:
  - Curvature defined by $F = dA + \frac{1}{2}[A, A]$.
  - Curvature measures deviation between M and G/H.

Example: Cartan geometry of spacetime

- Choose Lie groups:
  - Let

$$G = \begin{cases} \mathrm{SO}_0(4,1) & \Lambda > 0 \\ \mathrm{ISO}_0(3,1) & \Lambda = 0 \\ \mathrm{SO}_0(3,2) & \Lambda < 0 \end{cases}, \quad H = \mathrm{SO}_0(3,1)\,.$$

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- Choose Cartan connection:
  - $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{z}$ splits into direct sum.
  - Let $e \in \Omega^1(P, \mathfrak{z})$ be the solder form of $\tilde{\pi} : P \to M$.
  - Let $\omega \in \Omega^1(P, \mathfrak{h})$ be the Levi-Civita connection.

$\Rightarrow$ $A = \omega + e \in \Omega^1(P, \mathfrak{g})$ is a Cartan connection.

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$\Rightarrow$ Spacetime (M, g) can be reconstructed from Cartan geometry.

Example: Cartan geometry of observer space

- Choose Lie groups: [S. Gielen, D. Wise '12]
  - Let
$$G = \begin{cases} \mathrm{SO}_0(4,1) & \Lambda > 0 \\ \mathrm{ISO}_0(3,1) & \Lambda = 0 \\ \mathrm{SO}_0(3,2) & \Lambda < 0 \end{cases}, \quad K = \mathrm{SO}(3).$$

  $\Rightarrow$ Coset spaces G/K are the maximally symmetric observer spaces.
- Choose principal K-bundle:
  - Let (M, g) be a Lorentzian manifold.
  - Let O be the future unit timelike vectors on M.
  - Let P be the oriented time-oriented orthonormal frames on M.

  $\Rightarrow$ $\pi : P \to O$ is a principal K-bundle.
- Choose Cartan connection:
  - $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{z}$ splits into direct sum.
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The clock postulate

- Proper time along a curve in Lorentzian spacetime:

$$\tau = \int_{t_1}^{t_2} \sqrt{-g_{ab}(x(t))\dot{x}^a(t)\dot{x}^b(t)}dt\,.$$

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- Finsler geometry: use a more general length functional:

$$\tau = \int_{t_1}^{t_2} F(x(t),\dot{x}(t))dt\,.$$

- Finsler function $F : TM \to \mathbb{R}^+$.
- Parametrization invariance requires homogeneity:

$$F(x,\lambda y) = \lambda F(x,y) \quad \forall \lambda > 0\,.$$

Definition of Finsler spacetimes

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$$g^F_{ab}(x, y) = \frac{1}{2} \frac{\partial}{\partial y^a} \frac{\partial}{\partial y^b} F^2(x, y) \,.$$

- Unit vectors $y \in T_x M$ defined by

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⇒ Set $\Omega_x \subset T_xM$ of unit timelike vectors at $x \in M$.

- Ω_x contains a closed connected component $S_x \subseteq \Omega_x$.
- Causality: S_x corresponds to physical observers.

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Observer space

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- Complete $y = f_0$ to a frame f_i with $g^F_{ab}(x, y)f_i^a f_j^b = -\eta_{ij}$.
- Let P be the space of all observer frames.

$\Rightarrow$ $\pi : P \to O$ is a principal SO(3)-bundle.

- In general no principal $SO_0(3, 1)$-bundle $\tilde{\pi} : P \to M$.

Cartan connection - translational part

- Need to construct $A \in \Omega^1(P, \mathfrak{g})$.
- Recall that

$$\begin{array}{ccccc} \mathfrak{g} & = & \mathfrak{h} & \oplus & \mathfrak{z} \\ A & = & \omega & + & e \end{array}$$

$\Rightarrow$ Need to construct $\omega \in \Omega^1(P, \mathfrak{h})$ and $e \in \Omega^1(P, \mathfrak{z})$.

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- Definition of e: Use the *solder form*.
  - Let $w \in T_{(x,f)}P$ be a tangent vector.
  - Differential of the projection $\tilde{\pi} : P \to M$ yields $\tilde{\pi}_*(w) \in T_xM$.
  - View frame f as a linear isometry $f : \mathfrak{z} \to T_xM$.
  - Solder form given by $e(w) = f^{-1}(\tilde{\pi}_*(w))$.

Cartan connection - boost / rotational part

- Definition of ω:
  - Frames (x, f) and (x, f') related by generalized Lorentz transform. [C. Pfeifer, M. Wohlfarth '11]
  - Relation between f and f' defined by parallel transport on O.
  - Tangent vector $w \in T_{(x,f)}P$ "shifts" frame f by small amount.
  - Compare shifted frame with parallely transported frame.
  - Measure the difference using the original frame:

$$\Delta f_i^a = \epsilon f_j^a \omega^j{}_i(w)\,.$$

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- Choose parallel transport on O so that g^F is covariantly constant.
- Connection on Finsler geometry: Cartan linear connection.

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$\Rightarrow$ Frames f_i^a and $f_i^a + \Delta f_i^a$ are orthonormal wrt the same metric.

$\Rightarrow$ $\omega(w) \in \mathfrak{h}$ is an infinitesimal Lorentz transform.

Complete Cartan connection

- Translational part $e \in \Omega^1(P, \mathfrak{z})$:

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$$\omega^i{}_j = f^{-1}{}^i{}_a \left[df^a_j + f^b_j \left(dx^c F^a{}_{bc} + (dx^d N^c{}_d + df^c_0) C^a{}_{bc} \right) \right] \,.$$

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- Coefficients of Cartan linear connection:

$$N^a{}_b = \frac{1}{4}\bar{\partial}_b \left[g^{F\,aq} \left(y^p \partial_p \bar{\partial}_q F^2 - \partial_q F^2\right)\right] \,,$$

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$\Rightarrow$ $A = \omega + e$ is a Cartan connection on $\pi : P \to O$.

Fundamental vector fields

- Let $a = z^i \mathcal{Z}_i + \frac{1}{2} h^i{}_j \mathcal{H}_i{}^j \in \mathfrak{g}$.
- Define the vector field

$$\underline{A}(a) = z^i f_i^a \left(\partial_a - f_j^b F^c{}_{ab} \bar{\partial}_c^j \right) + \left(h^i{}_j f_i^a - h^i{}_0 f_i^b f_j^c C^a{}_{bc} \right) \bar{\partial}_a^j \,.$$

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$\Rightarrow$ For all $p \in P$ we find

$$A(\underline{A}(a)(p)) = a\,.$$

$\Rightarrow$ For all $w \in T_pP$ we find

$$\underline{A}(A(w))(p) = w\,.$$

$\Rightarrow$ $A_p : T_pP \to \mathfrak{g}$ and $\underline{A}_p : \mathfrak{g} \to T_pP$ complement each other.

Split of the tangent bundle TP

- Consider adjoint representation $\mathrm{Ad} : K \subset G \to \mathrm{Aut}(\mathfrak{g})$ of K on $\mathfrak{g}$.
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- $\mathfrak{g}$ splits into irreducible subrepresentations of Ad.
- Induced decompositions of A and TP:

$$\begin{array}{ccccccccc}
\mathfrak{g} & = & \mathfrak{k} & \oplus & \mathfrak{y} & \oplus & \vec{\mathfrak{z}} & \oplus & \mathfrak{z}0 \\
\uparrow A & = & \uparrow \Omega & + & \uparrow b & + & \uparrow \vec{e} & + & \uparrow e^0 \\
T_pP & = & R_pP & \oplus & B_pP & \oplus & \vec{H}_pP & \oplus & H^0_pP \\
 & & \Updownarrow & & \Updownarrow & & \Updownarrow & & \Updownarrow \\
 & & \text{rotations} & & \text{boosts} & & \text{spatial translations} & & \text{temporal translations}
\end{array}$$

- Subbundles of TP spanned by fundamental vector fields $\underline{A}$.

Time translation

- Consider the fundamental vector field

$$\mathbf{t} = \underline{A}(\mathcal{Z}_0) = f_0^a \partial_a - f_j^a N^b{}_a \bar{\partial}_b^j \qquad \Leftrightarrow \qquad \omega^i{}_j(\mathbf{t}) = 0\,, \quad e^i(\mathbf{t}) = \delta_0^i\,.$$

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$\Rightarrow$ Frame f is parallely transported.

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- $R^d{}_{cab}$, $P^d{}_{cab}$, $S^d{}_{cab}$: curvature of Cartan linear connection.

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$$\begin{array}{ccccccc}
R_pP & \oplus & B_pP & \oplus & H_pP & = & T_pP \\
\downarrow \pi_* & & \downarrow \pi_* & & \downarrow \pi_* & & \\
0 & & B_oO & \oplus & H_oO & = & T_oO \\
& & \downarrow \pi'_* & & \downarrow \pi'_* & & \\
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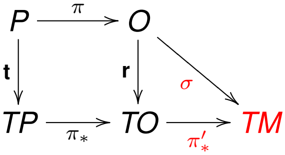

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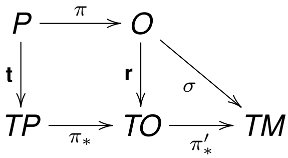

- Define the map $\sigma = \pi'_* \circ \mathbf{r}$.
- σ is in general not an embedding.
- Impose this as another condition.

Finsler geometry

- Finsler function must be positively homogeneous of degree one:

$$F(x, \lambda y) = |\lambda| F(x, y)$$

- Unit timelike condition: $F(\sigma(o)) = 1$ for all observers $o \in O$.

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- No Finsler geometry on $TM \setminus \mathbb{R}\sigma(O)$.
- Cartan geometry describes only geometry visible to observers.

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Reconstruction of a given Finsler spacetime

- Idea:
  - Start from a Finsler spacetime (M, F).
  - Construct a Cartan observer space $(\pi : P \to O, A)$.
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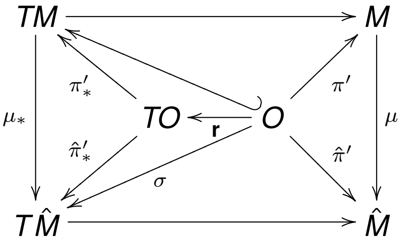

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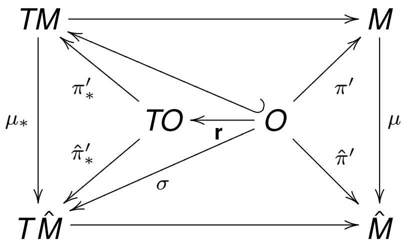

- μ preserves the Finsler function on timelike vectors.

$\Rightarrow$ Reconstruction of the original Finsler geometry.

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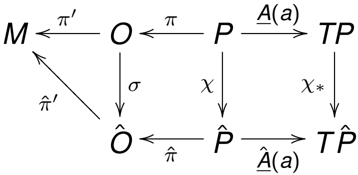

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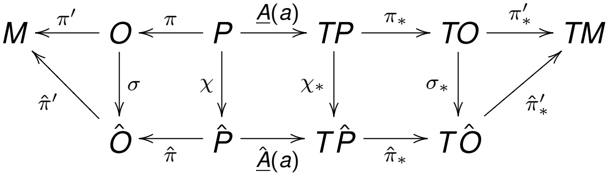

- Every Cartan morphism $\chi = (x, f)$ takes the form

$$x(p) = \pi'(\pi(p)), \quad f_i(p) = \pi'_*(\pi_*(\underline{A}(\mathcal{Z}_i)(p)))$$

$\Rightarrow$ Simple test for equivalence of $(\pi : P \to O, A)$ and $(\hat{\pi} : \hat{P} \to \hat{O}, \hat{A})$.

Outline

Gravity from Cartan to Finsler

- MacDowell-Mansouri gravity on observer space: [S. Gielen, D. Wise '12]

$$S_G = \int_O \epsilon_{\alpha\beta\gamma} \, \mathrm{tr}_{\mathfrak{h}}(F_{\mathfrak{h}} \wedge \star F_{\mathfrak{h}}) \wedge b^\alpha \wedge b^\beta \wedge b^\gamma$$

  - Hodge operator $\star$ on $\mathfrak{h}$.
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- Translate terms into Finsler language (with $R = d\omega + \frac{1}{2}[\omega, \omega]$):
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$$[e, e] \wedge \star R \rightsquigarrow g^{F\,ab} R^c{}_{acb}\, dV\,.$$

  - Cosmological constant:

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  - Study of exact solutions.
  - Effects of deviations from metric geometry?
  - Geometrodynamics of Finsler spacetimes.
  - ...