Observer space geometry of Finsler spacetimes

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Outline

Acknowledgements

- Cartan geometry of observer space:
  - Steffen Gielen
  - Derek Wise
- Finsler spacetimes:
  - Christian Pfeifer
  - Mattias Wohlfarth

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1. Physical motivation
2. Cartan geometry on observer space
3. Finsler spacetimes
4. Observer space of Finsler spacetimes
5. Conclusion

A simple experiment

- A supernova occurs in a far away galaxy.
- An astronomer points his telescope to the sky.
- He takes a picture of the supernova.

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- An astronomer points his telescope to the sky.
- He takes a picture of the supernova.
- How can we describe this experiment?
- What does it tell us about “spacetime”?

The spacetime picture

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  - Model spacetime as a Lorentzian manifold (M, g).
  - Supernova is a “beacon” at some event $x_0 \in M$.
  - Astronomer observes the light at another event $x \in M$.
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  - Tangent vector to γ in x determines direction of light propagation.
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- How much imagination does this picture require?

The actual measurement

- Fix an observer frame:
  - Location of the observer: spacetime event x.
  - Four-velocity of the observer: future timelike unit tangent vector f_0.
  - Coordinate axes of the observatory: spatial frame components f_i.

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$\Rightarrow$ “Beacon” event x_0 and geodesic γ are part of the interpretation.

Spacetime vs. observer space

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  - Spacetime geometry given by Lorentzian manifold (M, g).
  - P is the space of orthonormal frames of (M, g).
  - $\Rightarrow$ $\tilde{\pi} : P \to M$ is a principal $SO_0(3, 1)$-bundle.
  - $\Rightarrow$ Observers (x, f) and (x, f') at the same event are related by Lorentz transform.

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- The observer space idea:
  - Observers (x, f_0, f_i) and (x, f_0, f_i') are related by rotation.
  - $\Rightarrow$ Consider a principal SO(3)-bundle $\pi : P \to O$.
  - Describe experiments on *observer space* O.

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  - In general no (absolute) spacetime M.
  - Geometry on observer space O?

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Klein geometry

- A *Klein geometry* is a Lie group G with a closed subgroup $H \subset G$.
- $\pi : G \to G/H = Z$ is a principal H-bundle.
- Tangent spaces $T_zZ \cong \mathfrak{z} = \mathfrak{g}/\mathfrak{h}$.
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- Geometric structure on $\pi : G \to Z$ induced by multiplication:
  - G acts on itself by left translation:

$$L : G \times G \to G, (g, g') \mapsto L_g g' = gg' .$$

  - Left translation induces the *Maurer-Cartan form* $A \in \Omega^1(G, \mathfrak{g})$:

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- Z is a *homogeneous space* - all points “look the same”.
- How can we describe *inhomogeneous* geometries?

Cartan geometry

- Idea: “local” version of Klein geometry.
- A *Cartan geometry* modeled on a Klein geometry G/H is a principal H-bundle $\pi : P \to M$ together with a 1-form $A \in \Omega^1(P, \mathfrak{g})$ (the *Cartan connection*) such that:
    - For each $p \in P$, $A_p : T_pP \to \mathfrak{g}$ is a linear isomorphism.
    - $(R_h)^*A = \mathrm{Ad}(h^{-1}) \circ A \quad \forall h \in H$.
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- Example: Every Klein geometry $\pi : G \to G/H$ with the Maurer-Cartan form A on G is a flat ($F = 0$) Cartan geometry.

Cartan geometry of spacetime

- Let

$$G = \begin{cases} \mathrm{SO}_0(4,1) & \Lambda > 0 \\ \mathrm{ISO}_0(3,1) & \Lambda = 0 \\ \mathrm{SO}_0(3,2) & \Lambda < 0 \end{cases}, \quad H = \mathrm{SO}_0(3,1)\,.$$

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- $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{z}$ splits into H-modules under Ad.
- Let $e \in \Omega^1(P, \mathfrak{z})$ be the solder form of $\tilde{\pi} : P \to M$.
- Let $\omega \in \Omega^1(P, \mathfrak{h})$ be the Levi-Civita connection.

$\Rightarrow$ $A = \omega + e \in \Omega^1(P, \mathfrak{g})$ is a Cartan connection.

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- General relativity can be lifted to observer space. [S. Gielen, D. Wise '12]

Outline

The clock postulate

- Proper time along a curve in Lorentzian spacetime:

$$\tau = \int_{t_1}^{t_2} \sqrt{-g_{ab}(x(t))\dot{x}^a(t)\dot{x}^b(t)}dt\,.$$

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- Finsler geometry: use a more general length functional:

$$\tau = \int_{t_1}^{t_2} F(x(t), \dot{x}(t))dt\,.$$

- Finsler function $F : TM \to \mathbb{R}^+$.
- Parametrization invariance requires homogeneity:

$$F(x, \lambda y) = \lambda F(x, y) \quad \forall \lambda > 0\,.$$

Definition of Finsler spacetimes

- Finsler geometries suitable for spacetimes exist. [C. Pfeifer, M. Wohlfarth '11]

⇒ Notion of timelike, lightlike, spacelike tangent vectors.

- Finsler metric

$$g^F_{ab}(x,y) = \frac{1}{2}\frac{\partial}{\partial y^a}\frac{\partial}{\partial y^b}F^2(x,y)\,.$$

- Unit vectors $y \in T_xM$ defined by

$$F^2(x,y) = g^F_{ab}(x,y)y^ay^b = 1\,.$$

⇒ Set $\Omega_x \subset T_xM$ of unit timelike vectors at $x \in M$.

- Ω_x contains a closed connected component $S_x \subseteq \Omega_x$.

Physics on Finsler spacetimes

- Geodesic motion:
  - Point particle action on Finsler spacetime:

$$\tau = \int_{t_1}^{t_2} F(x(t), \dot{x}(t)) dt \,.$$

  - Finsler geodesics extremizing this action:

$$0 = \ddot{x}^a + N^a{}_b(x, \dot{x})\dot{x}^b \,.$$

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- Gravity:
  - Gravity action: [C. Pfeifer, M. Wohlfarth '11]

$$S_G = \int_\Sigma d^4x \, d^3y \, \sqrt{-\tilde{G}} R^a{}_{ab} y^b \,.$$

  - Unit tangent bundle $\Sigma = \{(x, y) \in TM | F(x, y) = 1\}$.
  - Sasaki metric $\tilde{G}$ on Σ.
  - Non-linear curvature $R^a{}_{ab}$.

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- Complete $y = f_0$ to a frame f_μ with $g^F_{ab}(x,y)f^a_\mu f^b_\nu = -\eta_{\mu\nu}$.
- Let P be the space of all observer frames.

$\Rightarrow$ $\pi : P \to O$ is a principal SO(3)-bundle.

Cartan connection - translational part

- Need to construct $A \in \Omega^1(P, \mathfrak{g})$.
- Recall that

$$\begin{array}{ccccc} \mathfrak{g} & = & \mathfrak{h} & \oplus & \mathfrak{z} \\ A & = & \omega & + & e \end{array}$$

$\Rightarrow$ Need to construct $\omega \in \Omega^1(P, \mathfrak{h})$ and $e \in \Omega^1(P, \mathfrak{z})$.

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- Definition of e: Use the *solder form*.
  - Let $w \in T_{(x,f)}P$ be a tangent vector.
  - Differential of the projection $\tilde{\pi} : P \to M$ yields $\tilde{\pi}_*(w) \in T_xM$.
  - View frame f as a linear isometry $f : \mathfrak{z} \to T_xM$.
  - Solder form given by $e(w) = f^{-1}(\tilde{\pi}_*(w))$.

Cartan connection - boost / rotational part

- Definition of ω:
  - Frames (x, f) and (x, f') related by generalized Lorentz transform.
    [C. Pfeifer, M. Wohlfarth '11]
  - Relation between f and f' defined by parallel transport on O.
  - Tangent vector $w \in T_{(x,f)}P$ "shifts" frame f by small amount.
  - Compare shifted frame with parallely transported frame.
  - Measure the difference using the original frame:

$$\Delta f^a_\mu = \epsilon f^a_\nu \omega^\nu{}_\mu(w) \,.$$

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- Choose parallel transport on O so that g^F is covariantly constant.
- Connection on Finsler geometry: Cartan linear connection.

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$\Rightarrow$ Frames f^a_μ and $f^a_\mu + \Delta f^a_\mu$ are orthonormal wrt the same metric.

$\Rightarrow$ $\omega(w) \in \mathfrak{h}$ is an infinitesimal Lorentz transform.

Complete Cartan connection

- Translational part $e \in \Omega^1(P, \mathfrak{z})$:

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- Coefficients of Cartan linear connection:

$$N^a{}_b = \frac{1}{4} \bar{\partial}_b \left[g^{F\,aq} \left(y^p \partial_p \bar{\partial}_q F^2 - \partial_q F^2 \right) \right] \,,$$

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$\Rightarrow$ $A = \omega + e$ is a Cartan connection on $\pi : P \to O$.

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- Boost / rotational part $F_{\mathfrak{h}} \in \Omega^2(P, \mathfrak{h})$:

$$d\omega^\mu{}_\nu + \omega^\mu{}_\rho \wedge \omega^\rho{}_\nu = -\frac{1}{2} f^{-1}{}^\mu_d f_\nu^c \Big(R^d{}_{cab} dx^a \wedge dx^b + 2P^d{}_{cab} dx^a \wedge \delta f_0^b + S^d{}_{cab} \delta f_0^a \wedge \delta f_0^b \Big) \,.$$

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with $\delta f^c_0 = dx^d N^c{}_d + df^c_0$.

- Boost / rotational part $F_{\mathfrak{h}} \in \Omega^2(P, \mathfrak{h})$:

$$d\omega^\mu{}_\nu + \omega^\mu{}_\rho \wedge \omega^\rho{}_\nu = -\frac{1}{2} f^{-1}{}^\mu_d f^c_\nu \Big(R^d{}_{cab} dx^a \wedge dx^b + 2P^d{}_{cab} dx^a \wedge \delta f^b_0 + S^d{}_{cab} \delta f^a_0 \wedge \delta f^b_0 \Big)\,.$$

- $R^d{}_{cab}$, $P^d{}_{cab}$, $S^d{}_{cab}$: curvature of Cartan linear connection.

Split of the tangent bundle TP

- $\mathfrak{g}$ splits into subrepresentations of $\mathrm{Ad} : K \subset G \to \mathrm{Aut}(\mathfrak{g})$:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{y} \oplus \vec{\mathfrak{z}} \oplus \mathfrak{z}_0 \,.$$

- Cartan connection $A \in \Omega^1(P, \mathfrak{g})$ splits:

$$A = \Omega + b + \vec{e} + e^0 \,.$$

  - Rotations: $\Omega \in \Omega^1(P, \mathfrak{k})$.
  - Boosts: $b \in \Omega^1(P, \mathfrak{y})$.
  - Spatial translations: $\vec{e} \in \Omega^1(P, \vec{\mathfrak{z}})$.
  - Temporal translation: $e^0 \in \Omega^1(P, \mathfrak{z}_0)$.
- Isomorphisms $A_p : T_pP \to \mathfrak{g}$ induce split of the tangent spaces:

$$T_pP = R_pP \oplus B_pP \oplus \vec{H}_pP \oplus H^0_pP \,.$$

Time translation

- Unique normalized section T of H^0P given by

$$\omega^{\mu}{}_{\nu}(T) = 0\,, \quad e^{\mu}(T) = \delta^{\mu}_{0}\,.$$

- Integral curve $\Gamma : \mathbb{R} \to P, t \mapsto (x(t), f(t))$ of T.

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- From $\omega^i{}_j(T) = 0$ follows:

$$0 = \dot{f}^a_i + f^b_i\left(\dot{x}^c F^a{}_{bc} + (\dot{x}^d N^c{}_d + \dot{f}^c_0)C^a{}_{bc}\right) = \nabla_{(\dot{x},\dot{f}_0)} f^a_i\,.$$

$\Rightarrow$ Frame f is parallely transported.

Reconstruction of spacetime (sketch)

- For $p \in P$, define the *vertical tangent space*

$$V_pP = R_pP \oplus B_pP = \{v \in T_pP | A(v) \in \mathfrak{h}\} .$$

- Vertical tangent bundle VP is a distribution on P.
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- Restrict charts to $Z_p = V_p \cap \mathfrak{z}$ to get homeomorphisms

$$\psi_p = \tilde{\pi} \circ \exp_p \circ \underline{A} : Z_p \subset \mathfrak{z} \to W_p \subset M .$$

$\Rightarrow$ Charts on M - hopefully C^∞.

Gravity (sketch)

- MacDowell-Mansouri gravity on observer space: [S. Gielen, D. Wise '12]

$$S = \int_O \mathrm{tr}_{\mathfrak{h}}(F_{\mathfrak{h}} \wedge \star F_{\mathfrak{h}}) \wedge \tau_{\mathfrak{y}}(b \wedge b \wedge b) + \dots$$

  - $\mathfrak{h}$-part of the curvature $F_{\mathfrak{h}}$.
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  - Non-degenerate H-invariant inner product $\mathrm{tr}_{\mathfrak{h}}$ on $\mathfrak{h}$.
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- Application to Finsler geometry (with $R = d\omega + \frac{1}{2}[\omega, \omega]$):
  - Curvature scalar:

$$[e, e] \wedge \star R \rightsquigarrow g^{F\,ab} R^c{}_{acb}\, dV\,.$$

  - Cosmological constant:

$$[e, e] \wedge \star [e, e] \rightsquigarrow dV\,.$$

  - Gauss-Bonnet term:

$$R \wedge \star R \rightsquigarrow \epsilon^{abcd} \epsilon^{efgh} R_{abef} R_{cdgh}\, dV\,.$$

Comparison with metric limit

- Finsler geometry of a Lorentzian manifold (M, g):

$$g^F_{ab} = -g_{ab}\,, \quad N^a{}_b = \Gamma^a{}_{bc} y^c\,, \quad F^a{}_{bc} = \Gamma^a{}_{bc}\,, \quad C^a{}_{bc} = 0\,.$$

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- Consider a section $f: M \to P$ of the frame bundle and $v \in T_xM$.
- $\mathfrak{h}$-part of the connection:

$$f^a_\nu \omega^\nu{}_\mu(f_*(v)) = v^b\partial_b f^a_\mu + v^c f^b_\mu \Gamma^a{}_{bc} = v^b\nabla_b f^a_\mu\,.$$

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- Curvature coefficients:

$$R^d{}_{cab} = \partial_b\Gamma^d{}_{ca} - \partial_a\Gamma^d{}_{cb} + \Gamma^e{}_{ca}\Gamma^d{}_{eb} - \Gamma^e{}_{cb}\Gamma^d{}_{ea}\,, \quad P^d{}_{cab} = S^d{}_{cab} = 0\,.$$

- Cartan curvature:

$$F_{\mathfrak{z}}{}^\mu = 0\,, \quad F_{\mathfrak{h}}{}^\mu{}_\nu = -\frac{1}{2} f^{-1}{}^\mu_d f^c_\nu R^d{}_{cab} dx^a \wedge dx^b\,.$$

$\Rightarrow$ The torsion $F_{\mathfrak{z}}$ vanishes and $F_{\mathfrak{h}}$ is the Riemannian curvature.

Outline

Summary

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Summary

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- Observer spaces:
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  - Describe observer space geometry using Cartan geometry.
- Observer space of Finsler spacetimes:
  - Finsler spacetimes possess well-defined observer space.
  - Cartan geometry on observer space derived from Finsler geometry.
  - Connection and curvature follow from Cartan linear connection.
  - Fermi-Walker transported frames given by the “flow of time”.

Outlook

- Current projects:
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- Future projects:
  - Consistent matter coupling.
  - Study of exact solutions.
  - Effects of deviations from metric geometry?
  - ...