Quantum aspects of spinning strings in $AdS_3 \times S^3 \times T^4$ with RR-flux

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VORWORT

Diese Arbeit widmet sich der Untersuchung eines wichtigen Beispiels der Dualität zwischen Eichtheorie und Stringtheorie, nämlich der AdS/CFT-Korrespondenz auf $AdS_3 \times S^3 \times T^4$ mit RR-Fluss. Das Hauptaugenmerk liegt dabei auf der Seite der Stringtheorie. Zunächst wird gezeigt, dass der klassische Superstring auf diesem Raum integrabel ist. Danach werden einige Lösungen mit rotierenden Strings konstruiert und ihre Energien sowohl klassisch als auch unter Berücksichtigung von Quantenkorrekturen berechnet. Die Kenntnis der String-Energien ist wichtig um die AdS/CFT-Korrespondenz zu testen, die eine Entsprechung der Energien von Strings und der anomalen Dimensionen von Operatoren in der Eichtheorie beinhaltet.

ABSTRACT

In this work we shall examine an important example for a duality between gauge theory and string theory, namely the AdS/CFT correspondence on $AdS_3 \times S^3 \times T^4$ with RR-flux. Attention will be payed mainly to the string theory side of the correspondence. Firstly, we will show that the classical superstring on this space is integrable. Secondly, we will construct several spinning string solutions and compute their energies at the classical level as well as their leading quantum corrections. The knowledge of the string energies is important for testing the AdS/CFT correspondence, which conjectures that the spectrum of string energies is identical to that of anomalous dimensions of planar gauge theory operators.

CONTENTS

1. INTRODUCTION

The original discovery that led to the development of string theory was the observation of a linear dependence between meson masses and their spins. This relation could be explained by Veneziano [1], using a model of a rotating, open string. Today, string theory has evolved to be one of the most promising candidates for a unified theory of gravity, quantum mechanics and all forces of nature. Its basic idea is to replace the notion of pointlike, elementary particles by one-dimensional, extended objects which can oscillate similar to a violin string - giving them their popular name.

Although string theory has more and more become a unified theory of gravity and quantum field theory than a pure theory of strong interactions, the original idea of writing a strongly coupled gauge theory in terms of a string theory is still of particular interest. The basic idea of relating gauge theory and string theory was developed by 't Hooft [2], who proposed that gauge theory and string theory provide holographically equivalent descriptions of the same problem. He conjectured that there exists an even deeper equivalence between these two types of theories, in a sense that they do not only lead to equivalent descriptions, but may be viewed as different aspects of the same theory.

The most explicit example for this equivalence has been introduced by Maldacena [3] in his famous conjecture of an equivalence between type IIB string theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ super yang mills theory in four dimensions. Several other equivalences relating string theories on anti-de-Sitter spacetimes and conformal field theories have been proposed, summarized under the notion of AdS/CFT correspondence. It relates the fundamental objects of both theories to each other, string states on the one side and gauge invariant operators on the other. An extensive review can be found in [4].

Since the AdS/CFT correspondence is a conjecture, the question arises how this conjecture might be tested. The most intuitive test is the comparison of string spectra with their gauge operator counterparts on the CFT side. An important difficulty here is that the conjecture relates the weakly coupled regime of string theory to the strongly coupled regime of CFT and vice versa, which makes a direct comparison of the two theories difficult. To circumvent this problem, it is fruitful to make use of the high amount of symmetry present on both sides of the conjecture. Considering states that are invariant under parts of the supersymmetry, so called BPS states, which thanks to non-renormalization theorems are protected, one can easily extrapolate from weak to strong coupling. However beyond the BPS spectrum, other means of comparing the two theories have to be found. What is potentially obstructing to such endeavors is the difficulty of quantizing string theory in $AdS_5 \times S^5$.

A very fruitful direction in testing the AdS/CFT correspondence, which has been initiated by the work of Berenstein, Maldacena, Nastase [16], Frolov, Tseytlin [?], Gubser, Klebanov, Polyakov [8], is to study a subclass of states with large quantum numbers. Consider e.g. the rotational symmetries of the S^5 component, corresponding to the R-symmetry of the CFT. On the string theory side, this leads to the classical limit of string solutions with high angular momentum, known as spinning strings. The advantage in considering these states is that semi-classical quantization can be used to compute quantum corrections to their classical energies. On the gauge theory side, the corresponding states are operators with a large number of field insertions, which can be represented by spin chains and thus can be investigated by techniques from condensed matter physics, known as the Bethe ansatz.

In the case of the AdS_5/CFT_4 correspondence much mileage has been gained by investigating the subclass of states with large angular momentum, which lead to strong support of the conjectured correspondence.

In this thesis, we will advance these methods in the case of AdS_3/CFT_2 correspondence. Our main focus is the string theory side, where we elaborate on the classical integrable structure. We shall investigate different spinning string solutions on $AdS_3 \times S^3 \times T^4$ and compute their classical energies as well as quantum corrections. The second chapter gives a short introduction into the topic of AdS/CFT correspondence and its possible tests, especially using spinning strings. In the third chapter we will state some geometrical properties of $AdS_3 \times S^3 \times T^4$ with RR-flux and derive the equations of motion and conserved charges from the superstring action. The fourth chapter covers classical integrability and the construction of flat currents. This establishes the classical integrability of the theory. Finally, the fifth chapter is devoted to quantum corrections to the energy of various spinning string configurations.

Appendix A lists the conventions and notation used in this thesis. The symmetry algebra $psu(1,1|2) \times \widetilde{psu}(1,1|2)$ and different bases of this algebra are reviewed in appendix B. A detailed derivation of invariant charges can be found in appendix C. Appendix D describes two Mathematica packages that have been written for this thesis.

2. ADS/CFT CORRESPONDENCE

2.1 Statement of the AdS/CFT correspondence

It has been proposed by 't Hooft [2] that, although string theory is quite different from gauge theory, there still exists a relationship between these two theories. The basic idea that led to this discovery was again the aim to gain a deeper understanding of QCD. 't Hooft suggested that the SU(N) theory might simplify if N is large, especially in the limit $N \to \infty$ the theory should be solvable. This would allow the $N = 3$ case to be solved by performing an expansion in $\frac{1}{N}$. We will show that the diagrammatic expansion of the field theory suggests that the large N theory is equivalent to a free string theory.

The idea of a gauge / string duality also applies to more general gauge theories. A particular class of gauge theories are theories, in which the gauge coupling does not depend on the energy scale. These theories are known to be conformally invariant. The conformal field theory, that has been investigated most intensively in the context of gauge / string duality, is $\mathcal{N} = 4$ super Yang-Mills theory. It has the maximal number of supersymmetry generators in four dimensions and its gauge group is SU(N). The theory contains the gauge fields (gluons) A_μ, four fermions (which can be written as a 16 component 10d Majorana-Weyl spinor $\chi_\alpha, \alpha = 1, \ldots, 16$) and six scalars $\phi_i, i = 1, \ldots, 6$. All fields transform in the adjoint representation of the gauge group. The Lagrangian is completely determined by supersymmetry and reads [5]

$$S = \frac{2}{g_{YM}^2} \int \mathrm{d}^4x \ \mathrm{Tr} \left(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} (D_\mu \phi_i)(D^\mu \phi_i) - \frac{1}{4} [\phi_i, \phi_j] [\phi_i, \phi_j] + \frac{1}{2} \bar{\chi} \not{D} \chi - \frac{i}{2} \bar{\chi} \Gamma_i [\phi_i, \chi] \right), \tag{2.1.1}$$

with the covariant derivative $D_\mu = \partial_\mu - i [A_\mu, .]$ and $\not{D} = D^\mu \Gamma_\mu$. The field strength $F_{\mu\nu}$ is defined as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. (Γ_μ, Γ_i) are the 10d gamma matrices. The only two parameters of the theory are the Yang-Mills coupling g_{YM} and the rank of the gauge group N. An important aspect is how to scale the coupling in the limit $N \to \infty$. A natural choice that has been motivated by QCD is the 't Hooft limit, scaling g_{YM} such that the 't Hooft coupling $\lambda = g_{YM}^2 N$ remains constant.

Besides the gauge symmetry, the theory has a global $\mathfrak{su}(4) \cong \mathfrak{so}(6)$ R-symmetry and the conformal symmetry with symmetry algebra $\mathfrak{so}(4,2)$ in four dimensions. It is thus intuitive to claim that these symmetries should also be found on the dual string theory side.

The most natural space that exhibits an SO$(4,2)$ symmetry is five dimensional anti-de-Sitter space AdS_5, which is the maximally symmetric Lorentzian space with constant negative curvature. Since superstring theory requires ten dimensions in order to be consistent, another five dimensions have to be added. The obvious choice is to put these dimensions into a five sphere S^5, since it also introduces the SO(6) symmetry into our theory. Furthermore the combined space is an exact solution to string theory. This vaguely suggests that $\mathcal{N} = 4$ super Yang-Mills theory is related to a superstring theory on $AdS_5 \times S^5$. It has been motivated by Maldacena [3] that the dual string theory of $\mathcal{N} = 4$ super Yang-Mills theory is indeed type IIB string theory on $AdS_5 \times S^5$.

In string theory, there are again two parameters: the string coupling g_s and the tension $\frac{1}{2\pi\alpha'}$. A third parameter, the common radius R of AdS_5 and S^5, can be absorbed by rescaling the embedding space coordinates. Type IIB string theory on $AdS_5 \times S^5$ involves the bosonic coordinates $x^m, m = 0, \ldots, 9$ of $AdS_5 \times S^5$ and two $D = 10$ Majorana-Weyl spinors $\theta^I, I = 1, 2$, with the dynamics given by an action of the form

$$S = \frac{1}{2\pi\alpha'} \int \mathrm{d}^2\sigma \left(\frac{R^2}{2} \sqrt{-g} g^{\mu\nu} G_{mn}(x) \partial_\mu x^m \partial_\nu x^n + i(\sqrt{-g} g^{\mu\nu} \delta^{IJ} - \epsilon^{\mu\nu} s^{IJ}) \bar{\theta}^I \rho_\mu D_\nu \theta^J + \ldots \right), \tag{2.1.2}$$

where $\mu, \nu = 1, 2$ labels the string coordinates, $s^{IJ} = \mathrm{diag}(1, -1)$ and higher terms have been omitted.

A motivation for the correspondence is given in [4] and will be briefly summarized here. Consider at first the gauge theory side and its perturbative expansion in terms of Feynman diagrams. Each diagram can be viewed as a tiling (or simplicial decomposition) of a two dimensional, oriented manifold. The numbers of vertices V, propagators E and loops F correspond to the numbers of corners, edges and surfaces of the tiling. To determine the powers of the 't Hooft coupling $\lambda = g_{YM}^2 N$ and N associated with this diagram, assume that each 3-point vertex carries a factor g_{YM} and each 4-point vertex carries a factor g_{YM}^2. The contribution of the vertices can thus be written as $g_{YM}^{n_3+2n_4}$, where n_3 is the number of 3-point vertices and n_4 is the number of 4-point vertices. Since $V = n_3 + n_4$ and $2E = 3n_3 + 4n_4$, we can also write this as g_{YM}^{2E-2V}. Finally, each loop contributes a factor N since we have to sum over N indices for each loop. It follows that the overall factor of a diagram has the form

$$g_{YM}^{2E-2V} N^F = \lambda^{E-V} N^{F-E+V} = \lambda^{E-V} N^{\chi} = \lambda^{E-V} N^{2-2g}, \tag{2.1.3}$$

where $F - E + V = \chi = 2 - 2g$ is the Euler characteristic of the manifold and g its genus. In the 't Hooft limit $N \to \infty, \lambda = \mathrm{const.}$ we find a diagrammatic expansion of the form

$$\sum_{g=0}^{\infty} N^{2-2g} \sum_{l=0}^{\infty} c_{g,l} \lambda^l = \sum_{g=0}^{\infty} N^{2-2g} f_g(\lambda). \tag{2.1.4}$$

In the limit $N \to \infty$, only diagrams with genus $g = 0$, so-called planar diagrams, contribute while all other diagrams are suppressed by a factor $\frac{1}{N^{2g}}$.

In perturbative string theory, a similar genus expansion exists. [5] A closer examination and comparison to the expansion derived above shows that there is also a correspondence between the parameters of the two theories, allowing us to identify

$$g_s = g_{YM}^2 = \frac{\lambda}{N}, \quad \alpha' = \frac{R^2}{\sqrt{\lambda}}. \tag{2.1.5}$$

From this we see that in the $N \to \infty$ limit, the string theory becomes non-interacting and string worldsheet corrections correspond to $\frac{1}{\sqrt{\lambda}}$ corrections.

Since the fundamental objects in CFT are gauge invariant operators, there should be some dual to these objects in string theory. The fundamental objects in string theory are strings, so it seems natural to propose that there exists a correspondence between gauge invariant operators and string states. Both objects transform in representations of a symmetry algebra, which is the conformal and R-symmetry algebra in the case of CFT and the isometry of $AdS_5 \times S^5$ in the case of string theory. Both algebras match, leading to the conclusion that dual objects in the sense of the correspondence should belong to the same representation of the symmetry algebra. Especially, the charges of both objects under the symmetry algebra should be identical. One of these charges is the energy E of the string, corresponding to the scaling dimension Δ of the gauge invariant operator on the CFT side. This leads to the conclusion that there should be a one to one correspondence between the energies of string states and the scaling dimensions of gauge invariant operators. In a similar fashion, the other conformal charges of the CFT are found to be corresponding to spins of AdS_5, whereas the R-symmetry charges correspond to S^5 spins on the string theory side.

Another motivation of the correspondence arises from superstring theory in flat Minkowski space with D-branes. Consider type IIB string theory in ten dimensional Minkowski space with N parallel D3-branes placed very close to each other. In this case there are two types of string excitations: open strings that end on the D-branes and closed strings, propagating through the bulk. In the low energy limit, only massless states can be excited, leading to an effective action

$$S = S_{\text{bulk}} + S_{\text{brane}} + S_{\text{int}}. \tag{2.1.6}$$

The bulk action describes the closed strings in the bulk and is given by type IIB supergravity. The brane action corresponds to the open string excitations of the D-branes, which are represented by maximally supersymmetric ($\mathcal{N} = 4$) super Yang-Mills theory in $3 + 1$ dimensions. Finally, S_{int} contains the interactions between open and closed strings. In the low energy limit, this coupling becomes weak and the brane modes decouple from supergravity.

On the other hand, one may view the D-branes as massive charged objects and, thus, sources of a supergravity field. Considering the low energy limit as in the previous picture leads once again to a decoupling of free type IIB supergravity from the remaining theory. But in this case the remaining theory is type IIB string theory in the near horizon region of the D-branes. More precisely, since the near horizon geometry of the D-branes is $AdS_5 \times S^5$ with RR-flux, the remaining theory is type IIB string theory on $AdS_5 \times S^5$.

The same theory, viewed from different points of view, decouples into type IIB supergravity on flat Minkowski space plus some extra theory, which is a conformal field theory in one case and a string theory on AdS space in the other case. This leads to the natural conclusion, that ($\mathcal{N} = 4$) super Yang-Mills theory in $3 + 1$ dimensions is equivalent, or dual, to IIB string theory on $AdS_5 \times S^5$ with RR-flux.

It is easy to construct similar correspondences. For example, one may consider a set of D$(p-2)$-branes instead of D3-branes. In this case the near horizon geometry is $AdS_p \times S^{10-p}$ and there is a different CFT which is conjectured to be dual to the string theory in this geometry. Another very well studied and very important case is that of $AdS_3 \times S^3 \times M_4$, where M_4 is some 4-dimensional, compact manifold. The most simple choice of M_4 is the 4-dimensional torus T^4, leading to the space $AdS_3 \times S^3 \times T^4$. [4] Unlike $AdS_3 \times S^3 \times M_4$ with NSNS flux this background cannot easily be studied by means of world-sheet CFT, although some progress in this direction has been obtained in [6, 7]. In this thesis, we will examine the case of a non-vanishing RR-flux in more detail.

2.2 Tests of the correspondence

Given the correspondence between $\mathcal{N}=4$ SYM theory and string theory on $AdS_5 \times S^5$, the question arises how to test this conjecture. The main problem in testing the correspondence is directly caused by the nature of the duality, which relates the weakly coupled regime of string theory to the strongly coupled regime of gauge theory and vice versa. Thus, if the states can be easily computed on one side of the correspondence, it is hard to find an approximation on the other site, which makes it difficult to compare those states and their counterparts.

A possible test of the correspondence arises from supersymmetry. States that are invariant under parts of the supersymmetry generators, named BPS states after Bogomol'nyi, Prasad and Sommerfeld, are not modified by quantum corrections and do not depend on the coupling constant. It is thus possible to compute them in the weak coupling regime and conclude that the spectrum is the same in the strong coupling regime, where it can be compared to the spectrum computed in the dual theory.

Another way to circumvent the problem of weak - strong coupling duality is to make use of the high amount of symmetry that is present on both sides of the correspondence. Both theories have the symmetry algebra $\mathfrak{su}(2,2) \oplus \mathfrak{su}(4) \cong \mathfrak{so}(2,4) \oplus \mathfrak{so}(6)$. In particular, these are found to be the charges of the conformal algebra and R-symmetry charges on the CFT side, which correspond the the AdS_5 spins and S^5 spins on the side of the string theory.

For testing the correspondence, it is useful to consider states where at least one of the R-symmetry charges (in the language of CFT), resp. S^5 spins (in the language of string theory) J is large. This allows a rescaling of the 't Hooft coupling by

$$\lambda' = \frac{\lambda}{J^2}. \tag{2.2.1}$$

The expansion of string energies / anomalous dimensions 2.1.3 then takes the form [9]

$$E = J\left(1 + \sum_{n=1}^{\infty} \lambda'^n \sum_{i=0}^{\infty} \frac{g_i^{(n)}}{J^i}\right). \tag{2.2.2}$$

The coefficients $g_i^{(n)}$ need to be determined separately on the string theory side and on the gauge theory side.

Since the CFT is weakly coupled when λ is small, one may perform an expansion in λ to compute the anomalous dimensions. Afterwards, considering the large J limit leads to an expansion in λ' as quoted above. This limit corresponds to a weakly coupled CFT in the limit of large spin, i.e. the limit of large gauge invariant operators.

On the string side of the correspondence, one can at first consider the limit $J \to \infty$ with λ' fixed to perform an expansion in $\frac{1}{J}$. In this limit, the string is rotating at high speed along the S^5 directions. The key point in considering this limit is that $\alpha' \sim \frac{1}{\sqrt{\lambda}} \sim \frac{1}{J}$ becomes small and thus quantum corrections to the classical solution can be computed semiclassically.

Both sides of the correspondence show that the same expansion in λ' and $\frac{1}{J}$ is possible, which allows us to compare not only the overall result, but also every single coefficient found in this expansion. Studying the behavior of both theories in the large J limit thus leads to a powerful test of the conjecture. A simple way to construct this kind of states will be explained in the following sections.

A similar treatment may be applied to the case of $AdS_3 \times S^3 \times T^4$. Since the symmetry algebra of $AdS_3 \times S^3$ is $\mathfrak{su}(1,1)^2 \oplus \mathfrak{su}(2)^2 \cong \mathfrak{so}(2,2) \oplus \mathfrak{so}(4)$, it is again possible to consider states large R-symmetry charges or S^3 spins, respectively.

2.3 Anomalous dimensions and Bethe ansatz

2.3.1 Anomalous dimensions and spin-chains

As mentioned in the previous section, it is fruitful to consider states carrying large charges. Since there are many possible ways to construct these states, it turns out to be useful to restrict oneself to an even smaller subset of states. We will present a short reminder of how such states can be constructed. For a detailed analysis, see [10].

Starting from $\mathcal{N} = 4$ super Yang-Mills theory, one may consider states that are built up from four of the six scalar fields, combined into two complex scalars:

$$Z = \phi_1 + i\phi_2, \quad W = \phi_3 + i\phi_4. \tag{2.3.1}$$

A simple way to construct local, gauge-invariant operators from these fields is given by

$$\mathcal{O} = \mathrm{Tr}\left(Z^{L-M} W^M + \text{permutations}\right), \tag{2.3.2}$$

where the terms in this sum are weighted by some phase factors. Since the theory is invariant under the conformal algebra, the states form representations of that algebra. In particular, they have to be eigenstates of the Casimir operators of the conformal algebra. One of them is the generator of scaling transformation, known as the dilatation operator. Up to one-loop order in perturbation theory, the dilatation operator acting on $\mathcal{O}$ is given by

$$D = L + \frac{\lambda}{16\pi^2} \sum_{l=1}^{L} (1 - \boldsymbol{\sigma}_l \cdot \boldsymbol{\sigma}_{l+1}) + O(\lambda^2) \tag{2.3.3}$$

with the set of Pauli matrices $\boldsymbol{\sigma}_l$ acting on the SU(2) doublet W, Z at position l. The leading order term, L, is the classical scaling dimension. It arises from the fact that the classical scaling dimension of each of the fields is 1 and the product consists of L fields in total. The correction to this classical scaling dimension is called the anomalous dimension and arises from quantum corrections.

As the previous formula suggests, there exists a relationship to a set of L spin-$\frac{1}{2}$ particles. One may reinterpret $\mathcal{O}$ as a spin chain with M spins down and $L - M$ spins up by the identifications $Z \leftrightarrow \uparrow$ and $W \leftrightarrow \downarrow$. The cyclicity of the trace implies that this spin chain is closed. In this picture, the one-loop dilatation operator acts as the Hamiltonian of a Heisenberg XXX spin chain with nearest neighbor interactions. Thus we have mapped the original problem to a problem with a well-known solution, that can be constructed using the Bethe ansatz.

The Bethe ansatz has the important property that many properties of the constructed states can be evaluated in the thermodynamic limit, i.e. in the limit $M \to \infty$ with $\frac{L}{M}$ fixed. [11] From this we may deduce that the Bethe ansatz is a good method for computing the anomalous dimensions of large gauge theory operators that arise in the context of AdS/CFT correspondence. It has to be tested whether this is actually the case. For the AdS_5 case, it has been shown that the Bethe ansatz does not reproduce the spectrum completely. [23] Thus, it is an important task to check how accurate the Bethe ansatz is on string backgrounds other than $AdS_5 \times S^5$ and whether it needs to be modified to compute the string spectrum.

2.3.2 The Bethe ansatz

An extensive review of the Bethe ansatz can be found in [11, 12, 13]. The original problem that has been solved by the Bethe ansatz method in 1931 is the Heisenberg model for a closed spin chain. In its most simple form, the Heisenberg model describes a linear chain of N spin-$\frac{1}{2}$ particles with nearest neighbor interactions. A useful basis of the Hilbert space is given by $|\sigma_1, \dots, \sigma_N\rangle, \sigma_i = \uparrow, \downarrow$. The Hamiltonian is given by

$$H = -J \sum_{n=1}^{N} \boldsymbol{S}_n \cdot \boldsymbol{S}_{n+1} = -J \sum_{n=1}^{N} \left(\frac{1}{2} \left(S_n^+ S_{n+1}^- + S_n^- S_{n+1}^+ \right) + S_n^z S_{n+1}^z \right), \tag{2.3.4}$$

where $\boldsymbol{S_n} = (S_n^x, S_n^y, S_n^z)$ is the spin operator at position n and $S_n^{\pm} = S_n^x \pm i S_n^y$ are the ladder operators with

$$S^z |\dots \uparrow \dots\rangle = \frac{1}{2} |\dots \uparrow \dots\rangle \qquad S^z |\dots \downarrow \dots\rangle = -\frac{1}{2} |\dots \downarrow \dots\rangle, \tag{2.3.5a}$$

$$S^+ |\dots \uparrow \dots\rangle = 0 \qquad S^+ |\dots \downarrow \dots\rangle = |\dots \uparrow \dots\rangle, \tag{2.3.5b}$$

$$S^- |\dots \uparrow \dots\rangle = |\dots \downarrow \dots\rangle \qquad S^- |\dots \downarrow \dots\rangle = 0. \tag{2.3.5c}$$

They satisfy the commutation relations

$$\left[S_m^i, S_n^j\right] = \epsilon_{ijk} \delta_{mn} S_m^k, \qquad \left[S_m^z, S_n^{\pm}\right] = \pm \delta_{mn} S_m^{\pm}, \qquad \left[S_m^+, S_n^-\right] = 2\delta_{mn} S_m^z \tag{2.3.6}$$

with $i, j, k = x, y, z$, which are the commutation relations of N sets of $\mathfrak{su}(2)$ generators.

An important property of this model is the fact that the total spin $\boldsymbol{S} = \sum_{n=1}^{N} \boldsymbol{S}_n$ commutes with the Hamiltonian, since

$$\begin{aligned} \left[H, S^j\right] &= -J \sum_{m,n=1}^{N} \left[S_n^i S_{n+1}^i, S_m^j\right] \\ &= -J \sum_{m,n=1}^{N} \epsilon_{ijk} \delta_{m,n+1} S_n^i S_{n+1}^k + \epsilon_{ijk} \delta_{m,n} S_n^k S_{n+1}^i \\ &= -J \sum_{n=1}^{N} \epsilon_{ijk} \left(S_n^i S_{n+1}^k + S_n^k S_{n+1}^i\right) = 0 \end{aligned} \tag{2.3.7}$$

As a consequence, the total spin is conserved. This allows us to choose the eigenstates of the Hamiltonian to be linear combinations of eigenstates of one spin component (say, S^z) with common eigenvalue. A basis with eigenstates of S^z thus puts H into block diagonal form and is the first step towards the Bethe ansatz.

The second symmetry that will be used is the symmetry under discrete translations. Let T be the operator that shifts the spin chain by one position, i.e.

$$T |\sigma_1, \dots, \sigma_N\rangle = |\sigma_2, \dots, \sigma_N, \sigma_1\rangle \tag{2.3.8}$$

From this we can deduce

$$T \boldsymbol{S}_n T^{-1} |\sigma_1, \dots, \sigma_N\rangle = T \boldsymbol{S}_n |\sigma_N, \sigma_1, \dots, \sigma_{N-1}\rangle = \boldsymbol{S}_{n+1} |\sigma_1, \dots, \sigma_N\rangle \tag{2.3.9}$$

and, since the basis is complete, $T\boldsymbol{S}_n T^{-1} = \boldsymbol{S}_{n+1}$. This leads to

$$THT^{-1} = -J\sum_{n=1}^{N} T\boldsymbol{S}_n \cdot \boldsymbol{S}_{n+1}T^{-1} = -J\sum_{n=1}^{N} T\boldsymbol{S}_n T^{-1} \cdot T\boldsymbol{S}_{n+1}T^{-1} = -J\sum_{n=1}^{N} \boldsymbol{S}_{n+1} \cdot \boldsymbol{S}_{n+2} = H \tag{2.3.10}$$

which implies $[H, T] = 0$. Similarly we find $[\boldsymbol{S}, T] = 0$, such that we can choose a basis in which both S^z and T are diagonal and H is block diagonal. This basis is the second step towards the Bethe ansatz.

In the following we will present the Bethe ansatz in its most general form and then restrict ourselves to two simple examples. Let

$$|n_1, \ldots, n_r\rangle = S^-_{n_1} \ldots S^-_{n_r} |\uparrow \ldots \uparrow\rangle, 1 \leq n_1 < \ldots < n_r \leq N \tag{2.3.11}$$

be a state with r spins pointing down. Obviously it is an eigenstate of S^z with $S^z |n_1, \ldots, n_r\rangle = \left(\frac{N}{2} - r\right) |n_1, \ldots, n_r\rangle$. Every linear combination of these states with a common number of down spins is another eigenstate of S^z with the same eigenvalue. We now consider

$$|\psi\rangle = \sum_{1 \leq n_1 < \ldots < n_r \leq N} a(n_1, \ldots, n_r) |n_1, \ldots, n_r\rangle \tag{2.3.12}$$

with

$$a(n_1, \ldots, n_r) = \sum_{\mathcal{P} \in \mathcal{S}_r} \exp\left(i\sum_{j=1}^{r} k_{\mathcal{P}j} n_j + \frac{i}{2}\sum_{1 \leq i < j \leq r} \theta_{\mathcal{P}i\mathcal{P}j}\right) \tag{2.3.13}$$

which is known as the Bethe ansatz. $k_i, i = 1, \ldots, r$ and $\theta_{ij} = -\theta_{ji}$ are called the momenta and phase angles, respectively. It is easy to check that $|\psi\rangle$ is an eigenstate of the translation operator T. Since

$$T|n_1, \ldots, n_r\rangle = TS^-_{n_1} \ldots S^-_{n_r} |\uparrow \ldots \uparrow\rangle = S^-_{n_1+1} \ldots S^-_{n_r+1} T |\uparrow \ldots \uparrow\rangle = |n_1 + 1, \ldots, n_r + 1\rangle \tag{2.3.14}$$

we have

$$T|\psi\rangle = \sum_{1 \leq n_1 < \ldots < n_r \leq N} a(n_1 - 1, \ldots, n_r - 1) |n_1, \ldots, n_r\rangle \tag{2.3.15}$$

Finally,

$$\begin{aligned} a(n_1 - 1, \ldots, n_r - 1) &= \sum_{\mathcal{P} \in \mathcal{S}_r} \exp\left(i\sum_{j=1}^{r} k_{\mathcal{P}j}(n_j - 1) + \frac{i}{2}\sum_{1 \leq i < j \leq r} \theta_{\mathcal{P}i\mathcal{P}j}\right) \\ &= \exp\left(-i\sum_{j=1}^{r} k_j\right) a(n_1, \ldots, n_r) \\ &= e^{-ik} a(n_1, \ldots, n_r) \end{aligned} \tag{2.3.16}$$

where we introduced the wave number $k = \sum_{j=1}^{r} k_j$. Thus we find $T|\psi\rangle = e^{-ik}|\psi\rangle$. The periodic boundary conditions require $a(n_1, \ldots, n_r) = a(n_2, \ldots, n_r, n_1 + N)$ which leads to

$$e^{ik_i N} = \exp\left(i\sum_{j \neq i} \theta_{ij}\right) \tag{2.3.17}$$

After taking logarithms and introducing the Bethe quantum numbers $\lambda_1, \ldots, \lambda_r \in \{0, 1, \ldots, N-1\}$ we end up with

$$Nk_i = 2\pi\lambda_i + \sum_{j\neq i} \theta_{ij} \tag{2.3.18}$$

It can be shown [11] that $|\psi\rangle$ is an eigenstate of H if

$$e^{i\theta_{ij}} = -\frac{e^{i(k_i+k_j)} + 1 - 2e^{ik_i}}{e^{i(k_i+k_j)} + 1 - 2e^{ik_j}} \tag{2.3.19}$$

or, equivalently,

$$2\cot\frac{\theta_{ij}}{2} = \cot\frac{k_i}{2} - \cot\frac{k_j}{2} \tag{2.3.20}$$

Equations 2.3.18 and 2.3.19 are known as the Bethe ansatz equations.

This method can be used to compute 1-loop anomalous dimensions, as we explained earlier. [10, 14, 15] Here we need to consider the thermodynamic limit of the spin chain, i.e. the case $N \to \infty$ where the spin flip ratio $\frac{r}{N}$ (which corresponds to the magnetization in condensed matter physics) remains fixed. The Bethe ansatz can also be evaluated in this limit.

2.4 Spinning strings

2.4.1 General method

We have seen that there exists a way to construct eigenstates of the dilatation operator on the gauge theory side, which are built from gauge invariant operators with a large number of field insertions. Thus one may ask which are the corresponding states on the string theory side. Many different solutions have been constructed and it turned out that the string theory duals are solutions with large quantum numbers, which are known as spinning strings. [5]

2.4.2 $AdS_5 \times S^5$ case

As already mentioned before, the symmetry algebra of $AdS_5 \times S^5$ is $\mathfrak{so}(2,4) \oplus \mathfrak{so}(6) \cong \mathfrak{su}(2,2) \oplus \mathfrak{su}(4)$, where the symmetry algebra $\mathfrak{so}(2,4) \cong \mathfrak{su}(2,2)$ of the AdS_5 component corresponds to the conformal symmetry of the gauge theory and the $\mathfrak{so}(6) \cong \mathfrak{su}(4)$ symmetry of the S^5 component corresponds to the R-symmetry. The conserved charges, which are the representation labels belonging to the generators of the Cartan subalgebra, are the energy E, two AdS_5 spins S_1, S_2 and three S^5 spins J_1, J_2, J_3.

One example is the plane-wave limit. In this case one considers a point-like string with infinite spin J. The effective background is then exactly solvable [16, 17]. More general spinning string solutions can be found, where the string is not point-like, but extends along various directions in $AdS_5 \times S^5$. Depending on the string configuration, one or more of the spin quantum numbers S_1, S_2, J_1, J_2, J_3 are large. The energy E of the string may then be written as a function $E(S_1, S_2, J_1, J_2, J_3)$ of the string quantum numbers.

2.4.3 $AdS_3 \times S^3 \times T^4$ case

On $AdS_3 \times S^3 \times T^4$, the situation is similar to the $AdS_5 \times S^5$ case. Here, the dual gauge theory is a two-dimensional CFT, which is the N'th symmetrical product of the T^4 CFT. The symmetry algebra is composed of $\mathfrak{su}(1,1)^2 \cong \mathfrak{so}(2,2)$ (the conformal symmetry of the CFT, corresponding to the symmetry of AdS_3), $\mathfrak{su}(2)^2 \cong \mathfrak{so}(4)$ (the R-symmetry of the gauge theory, corresponding to the symmetry of S^3) and four copies of $\mathfrak{u}(1)$ for the T^4 directions. The conserved charges given by the generators of the Cartan subalgebra are the energy E, the AdS_3 spins S, two S^3 spins J_1, J_2 and four T^4 momenta P_1, P_2, P_3, P_4.

There are three solutions that will be of particular interest in this thesis. For the first solution, the non-vanishing charges (besides the energy) are the two spins S and J_1, such that the string effectively rotates in an $AdS_3 \times S^1$ subspace with the S^1 component embedded in S^3. An obvious variation of this solution is given by embedding the S^1 in T^4 instead of S^3. In this case, J_1 vanishes, while the string acquires a non-vanishing T^4 momentum P_1. Both solutions correspond to the $\mathfrak{su}(1,1)$ sector of the dual CFT. In contrast, one may consider a solution corresponding to the $\mathfrak{su}(2)$ sector. This solution has two non-vanishing S^3 spins J_1, J_2 and effectively rotates in $\mathbb{R} \times S^3$. Only the solution with equal spins $J' = J_1 = J_2$ will be investigated in detail.

3. STRINGS ON $ADS_3 \times S^3 \times T^4$

3.1 The supergeometry of $AdS_3 \times S^3 \times T^4$ with RR-flux

The superspace examined in this thesis is composed of three-dimensional anti-de-Sitter space AdS_3, a three-sphere S^3 and a four-torus T^4, along with 16 fermionic degrees of freedom. The bosonic part of this space can be canonically embedded into $\mathbb{R}^{1,3}\times\mathbb{R}^4\times T^4$, which greatly simplifies the parametrization. The fact that the motion of the string is restricted to the embedded $AdS_3 \times S^3 \times T^4$ subspace will be incorporated later by introducing Lagrange multipliers into the string action. The coordinates of $\mathbb{R}^{1,3}$, $\mathbb{R}^4$ and T^4 will be denoted $\mathrm{Y}_S, S = 0,\ldots,3$, $\mathrm{X}_P, P = 1,\ldots,4$ and $Z_v, v = 1,\ldots,4$. It is convenient to use complex coordinates $Y_s, s = 0,1$ and $X_p, p = 1,2$, which are defined as

$$Y_0 = \mathrm{Y}_3 + i\mathrm{Y}_0, \qquad Y_1 = \mathrm{Y}_1 + i\mathrm{Y}_2, \tag{3.1.1a}$$

$$X_1 = \mathrm{X}_1 + i\mathrm{X}_2, \qquad X_2 = \mathrm{X}_3 + i\mathrm{X}_4. \tag{3.1.1b}$$

We finally end up with the following situation:

$$\begin{array}{ccccc} AdS_3 & \times & S^3 & \times & T^4 \\ \cap & & \cap & & \cap \\ \mathbb{R}^{1,3} & \times & \mathbb{R}^4 & \times & T^4 \\ \Cup & & \Cup & & \Cup \\ Y_s & & X_p & & Z_v \end{array} \tag{3.1.2}$$

The metric reads

$$\mathrm{d}s^{\,2} = \mathrm{d}Y_s^* \;\mathrm{d}Y^s \;+\; \mathrm{d}X_p^* \;\mathrm{d}X_p \;+\; \mathrm{d}Z_v \;\mathrm{d}Z_v \tag{3.1.3}$$

Here we introduced $Y^s = \eta^{st}Y_t$ with $\eta^{st} = \mathrm{diag}(-+)$. A common parametrization of $AdS_3 \times S^3$ is given by

$$Y_0 = e^{it}\cosh\rho, \qquad Y_1 = e^{i\chi}\sinh\rho \tag{3.1.4a}$$

$$X_1 = e^{i\psi}\cos\gamma, \qquad X_2 = e^{i\phi}\sin\gamma \tag{3.1.4b}$$

Using these coordinates, we find the metric

$$\mathrm{d}s^{\,2} = -\,\mathrm{d}t^{\,2}\cosh^2\rho + \mathrm{d}\chi^{\,2}\sinh^2\rho + \mathrm{d}\rho^{\,2} + \mathrm{d}\psi^{\,2}\cos^2\gamma + \mathrm{d}\phi^{\,2}\sin^2\gamma + \mathrm{d}\gamma^{\,2} + \mathrm{d}Z_{\,v}\,\mathrm{d}Z_{\,v} \tag{3.1.5}$$

The set $t,\chi,\rho,\psi,\phi,\gamma,Z_1,\ldots,Z_4$ of bosonic $AdS_3 \times S^3 \times T^4$ coordinates will be denoted by $\mathcal{X}_m, m = 0\ldots 9$. In addition, we need a set of fermionic coordinates $\theta^I, I = 1,2$, where θ^I is a 10d Majorana-Weyl spinor.

3.2 The superstring action

The complete superstring action can be found in [18]. It can be written in the form

$$I = \sqrt{\lambda} \int \mathrm{d}\tau \int_0^{2\pi} \frac{\mathrm{d}\sigma}{2\pi} \sqrt{-g}(\mathcal{L}_B + \mathcal{L}_F) \tag{3.2.1}$$

where $\mathcal{L}_B$ denotes the bosonic and $\mathcal{L}_F$ the fermionic part of the Lagrangian. Using the superspace coordinates that were introduced in the previous section we can write the bosonic part as

$$\mathcal{L}_B = -\frac{1}{2} g^{\mu\nu} \left(\partial_\mu X_p^* \partial_\nu X_p + \partial_\mu Y_s^* \partial_\nu Y^s + \partial_\mu Z_v \partial_\nu Z_v\right) + \frac{1}{2}\left(\Lambda(X_p^* X_p - 1) + \tilde{\Lambda}(Y_s^* Y^s + 1)\right) \tag{3.2.2}$$

The Lagrange multipliers $\Lambda, \tilde{\Lambda}$ have been introduced to restrict the motion of the string to the embedded $AdS_3 \times S^3$ space by imposing the constraints

$$X_p^* X_p - 1 = Y_s^* Y^s + 1 = 0 \tag{3.2.3}$$

The fermionic part of the Lagrangian is more involved and only the part $\mathcal{L}_F^{(2)}$ that is quadratic in the fermion fields is required in this work. It can be written as

$$\mathcal{L}_F^{(2)} = i(\eta^{\mu\nu}\delta^{IJ} - \epsilon^{\mu\nu} s^{IJ})\bar{\theta}^I \rho_\mu D_\nu \theta^J \tag{3.2.4}$$

with $s^{IJ} = \mathrm{diag}(1, -1)$. Here we used the projected Dirac matrices $\rho_\mu = \Gamma_M e_\mu^M$, where $e_\mu^M = E_m^M(\mathcal{X})\partial_\mu \mathcal{X}^m$ is the projected vielbein and E_m^M is the $AdS_3 \times S^3 \times T^4$ vielbein

$$E_m^M = \mathrm{diag}(\cosh\rho, 1, \sinh\rho, 1, \cos\gamma, \sin\gamma, 1, 1, 1, 1) \tag{3.2.5}$$

The covariant derivative is defined as

$$D_\mu \theta^I = \left(\delta^{IJ}\left(\partial_\mu + \frac{1}{4}\omega_\mu^{MN}\Gamma_{MN}\right) + \frac{i}{4}\epsilon^{IJ} e_\mu^A \Gamma_0\Gamma_1\Gamma_2\Gamma_A\right)\theta^J \tag{3.2.6}$$

Here ω_μ^{MN} denotes the Lorentz connection $\omega_\mu^{MN} = \partial_\mu \mathcal{X}^m \omega_m^{MN}$. The only non-vanishing components of ω_m^{MN} (up to symmetry) are

$$\omega_0^{01} = \sinh\rho, \quad \omega_2^{12} = -\cosh\rho, \quad \omega_4^{34} = \sin\gamma, \quad \omega_5^{35} = -\cos\gamma \tag{3.2.7}$$

3.3 Equations of motion

The classical equations of motion can be computed by variation of the string action 3.2.1. Let us consider the purely bosonic fields. Variation of the Lagrange multipliers Λ, Λ' yields the constraint equations

$$X_p^* X_p - 1 = Y_s^* Y^s + 1 = 0 \tag{3.3.1}$$

which is, of course, what we expect. Variation of the fields X_p, Y_s, Z_v leads to

$$\partial_\mu\left(\sqrt{-g} g^{\mu\nu}\partial_\nu X_p\right) + \Lambda X_p = \partial_\mu\left(\sqrt{-g} g^{\mu\nu}\partial_\nu Y_s\right) + \tilde{\Lambda} Y_s = \partial_\mu\left(\sqrt{-g} g^{\mu\nu}\partial_\nu Z_v\right) = 0 \tag{3.3.2}$$

Finally, the metric $g^{\mu\nu}$ has to be varied, leading to the Virasoro constraints

$$\partial_\mu X_p^* \partial_\nu X_p + \partial_\mu Y_s^* \partial_\nu Y^s + \partial_\mu Z_v \partial_\nu Z_v - \frac{1}{2} g_{\mu\nu} g^{\rho\sigma}\left(\partial_\rho X_p^* \partial_\sigma X_p + \partial_\rho Y_s^* \partial_\sigma Y^s + \partial_\rho Z_v \partial_\sigma Z_v\right) = 0 \tag{3.3.3}$$

In conformal gauge, the last two equations take the simple and well-known form

$$(\Box + \Lambda)X_p = (\Box + \tilde{\Lambda})Y_s = \Box Z_v = 0 \tag{3.3.4}$$

and

$$\dot{X}_p^* \dot{X}_p + \dot{Y}_s^* \dot{Y}^s + \dot{Z}_v \dot{Z}_v + X_p'^* X_p' + Y_s'^* Y^{s\prime} + Z_v' Z_v' = \dot{X}_p^* X_p' + \dot{Y}_s^* Y^{s\prime} + \dot{Z}_v Z_v' = 0. \tag{3.3.5}$$

Here we introduced the abbreviations

$$\dot{} = \frac{\partial}{\partial \tau} \quad \text{and} \quad ' = \frac{\partial}{\partial \sigma} \tag{3.3.6}$$

for the derivatives with respect to the parameters of the string worldsheet.

3.4 Conserved charges

Writing the bosonic part of the action 3.2.2 in terms of the coordinates $\mathcal{X}_m$, we find

$$\begin{aligned} \mathcal{L}_B = -\frac{1}{2} g^{\mu\nu} (&-\partial_\mu t \partial_\nu t \cosh^2 \rho + \partial_\mu \chi \partial_\nu \chi \sinh^2 \rho + \partial_\mu \rho \partial_\nu \rho \\ &+ \partial_\mu \psi \partial_\nu \psi \cos^2 \gamma + \partial_\mu \phi \partial_\nu \phi \sin^2 \gamma + \partial_\mu \gamma \partial_\nu \gamma + \partial_\mu Z_v \partial_\nu Z_v) \end{aligned} \tag{3.4.1}$$

From this, we see that t, χ, ψ, ϕ, Z_v are cyclic coordinates, i.e. the Lagrangian is invariant under the transformations $t \to t + \Delta t, \chi \to \chi + \Delta\chi, \psi \to \psi + \Delta\psi, \phi \to \phi + \Delta\phi, Z_v \to Z_v + \Delta Z_v$. According to the Noether theorem, each of them corresponds to a conserved quantity. In our case, the conserved charges are the energy E, AdS_3 spin S, S^3 spins J_1, J_2 and T^4 momenta P_v. They are given by

$$E = \sqrt{\lambda} \int_0^{2\pi} \frac{d\sigma}{2\pi} \dot{t} \cosh^2 \rho, \tag{3.4.2a}$$

$$S = \sqrt{\lambda} \int_0^{2\pi} \frac{d\sigma}{2\pi} \dot{\chi} \sinh^2 \rho, \tag{3.4.2b}$$

$$J_1 = \sqrt{\lambda} \int_0^{2\pi} \frac{d\sigma}{2\pi} \dot{\psi} \cos^2 \gamma, \tag{3.4.2c}$$

$$J_2 = \sqrt{\lambda} \int_0^{2\pi} \frac{d\sigma}{2\pi} \dot{\phi} \sin^2 \gamma, \tag{3.4.2d}$$

$$P_v = \sqrt{\lambda} \int_0^{2\pi} \frac{d\sigma}{2\pi} \dot{Z}_v. \tag{3.4.2e}$$

The conserved charges, divided by their common factor $\sqrt{\lambda}$, will be denoted $\mathcal{E}, \mathcal{S}, \mathcal{J}_1, \mathcal{J}_2, \mathcal{P}_v$, respectively.

4. CLASSICAL INTEGRABILITY

In this chapter we shall show that the string on $AdS_3 \times S^3 \times T^4$ with RR flux is classically integrable. This is done by constructing an infinite set of commuting charges. This approach is similar to [19], where two one-parameter sets of flat currents have been constructed, i.e. one-forms a satisfying

$$\mathrm{d}a + a \wedge a = 0. \tag{4.0.1}$$

4.1 The $psu(1,1|2) \times \widetilde{psu}(1,1|2)$ sigma model

The symmetry algebra of the $AdS_3 \times S^3$ with RR 3-form background may be represented as a direct sum of two copies of $psu(1,1|2)$ superalgebra, i.e. as $\mathcal{G} := psu(1,1|2) \oplus \widetilde{psu}(1,1|2)$. [18] This algebra $psu(1,1|2)$ is generated by the $(2|2) \times (2|2)$ supermatrices

$$\left(\begin{array}{c|c} m^{\alpha}{}_{\beta} & q^{\alpha}_{\beta'} \\ \hline q^{\alpha'}_{\beta} & m^{\alpha'}{}_{\beta'} \end{array}\right) \tag{4.1.1}$$

with $m^{\alpha}{}_{\beta} \in \mathfrak{su}(1,1)$, $m^{\alpha'}{}_{\beta'} \in \mathfrak{su}(2)$ and 8 fermionic generators $q^{\alpha}_{\alpha'}, q^{\beta'}_{\beta}$. For other bases, see appendix B.

The purely bosonic part of the algebra $psu(1,1|2)$ is $\mathfrak{su}(1,1) \oplus \mathfrak{su}(2)$. It is generated by the bosonic elements $m^{\alpha}{}_{\beta}$, $m^{\alpha'}{}_{\beta'}$. Since $\mathcal{G}$ contains two copies of $psu(1,1|2)$, the bosonic subalgebras can be grouped as $\mathfrak{su}(1,1)^2 \oplus \mathfrak{su}(2)^2 \cong \mathfrak{so}(2,2) \oplus \mathfrak{so}(4)$. Thus we obtain the bosonic symmetry algebra of $AdS_3 \times S^3$.

An important property of the algebra $\mathcal{G}$ is the fact that it obeys a $\mathbb{Z}_4$ grading, i.e. the algebra can be decomposed into four subspaces

$$\mathcal{G} = \mathcal{G}_0 + \mathcal{G}_1 + \mathcal{G}_2 + \mathcal{G}_3 \tag{4.1.2}$$

and the equation

$$[\mathcal{G}_m, \mathcal{G}_n\} \subseteq \mathcal{G}_{m+n} \tag{4.1.3}$$

holds for $m, n \in \mathbb{Z}_4$. Here $[.,.\}$ denotes the anticommutator between two fermionic operators and the commutator otherwise.

4.2 Flat currents

Our construction of flat currents will be similar to [19] for $AdS_5 \times S^5$ and we are using the basis stated in B.1. We start with a two-dimensional sigma model with the target space

$\mathrm{SU}(1,1|2)^2/(\mathrm{SO}(1,2)\times\mathrm{SO}(3))$. Let G denote the Lie supergroup of $\mathcal{G}$ and $g(x)$ a field taking values in G. From the Lagrangian $L \propto \mathrm{Tr}(\partial_i g^{-1}\partial^i g)$ we obtain a global left and right multiplication symmetry. The conserved current $L = g^{-1}\,\mathrm{d}g$ corresponding to right multiplication takes values in $\mathcal{G}$. Writing L as a one-form, we see that

$$\mathrm{d}L + L\wedge L = 0 \tag{4.2.1}$$

Using the Basis stated in B.1, we can write the current as

$$L = L^{\hat{a}}P_{\hat{a}} + \frac{1}{2}L^{\hat{a}\hat{b}}J_{\hat{a}\hat{b}} + \frac{1}{2}\bar{L}^I Q^I + \frac{1}{2}\bar{Q}^I L^I \tag{4.2.2}$$

Since the algebra has a $\mathbb{Z}_4$ grading with

$$\mathcal{G}_0 = \{J_{ab}, J_{a'b'}\},\quad \mathcal{G}_1 = \{Q^1,\bar{Q}^1\},\quad \mathcal{G}_2 = \{P_a, P_{a'}\},\quad \mathcal{G}_3 = \{Q^2,\bar{Q}^2\} \tag{4.2.3}$$

we can decompose the current according to the grading

$$H = -\frac{1}{2}L^{\hat{a}\hat{b}}J_{\hat{a}\hat{b}},\quad P = -L^{\hat{a}}P_{\hat{a}},\quad Q^I = -\frac{1}{2}\bar{L}^I Q^I - \frac{1}{2}\bar{Q}^I L^I \tag{4.2.4}$$

By construction, L satisfies the Maurer Cartan equation 4.2.1. This equation can be decomposed according to the grading, which leads to

$$\begin{aligned}
\mathrm{d}H &= H\wedge H + P\wedge P + Q^1\wedge Q^2 + Q^2\wedge Q^1 &(4.2.5a)\\
\mathrm{d}P &= H\wedge P + P\wedge H + Q^1\wedge Q^1 + Q^2\wedge Q^2 &(4.2.5b)\\
\mathrm{d}Q^1 &= H\wedge Q^1 + Q^1\wedge H + P\wedge Q^2 + Q^2\wedge P &(4.2.5c)\\
\mathrm{d}Q^2 &= H\wedge Q^2 + Q^2\wedge H + P\wedge Q^1 + Q^1\wedge P &(4.2.5d)
\end{aligned}$$

After a basis transformation

$$Q = Q^1 + Q^2,\quad Q' = Q^1 - Q^2 \tag{4.2.6}$$

the Maurer Cartan equations translate to

$$\begin{aligned}
\mathrm{d}H &= H\wedge H + P\wedge P + \frac{1}{2}(Q\wedge Q - Q'\wedge Q') &(4.2.7a)\\
\mathrm{d}P &= H\wedge P + P\wedge H + \frac{1}{2}(Q\wedge Q + Q'\wedge Q') &(4.2.7b)\\
\mathrm{d}Q &= H\wedge Q + Q\wedge H + P\wedge Q + Q\wedge P &(4.2.7c)\\
\mathrm{d}Q' &= H\wedge Q' + Q'\wedge H + P\wedge Q' + Q'\wedge P &(4.2.7d)
\end{aligned}$$

To construct the flat currents, we use the lowercase forms defined by $h = gHg^{-1}, p = gPg^{-1}, q = gQg^{-1}, q' = gQ'g^{-1}$. Expressing the Maurer Cartan equations in terms of the lowercase forms yields

$$\begin{aligned}
\mathrm{d}h &= -h\wedge h + p\wedge p - (h\wedge p + p\wedge h) - (h\wedge q + q\wedge h) + \frac{1}{2}(q\wedge q - q'\wedge q') &(4.2.8a)\\
\mathrm{d}p &= -2p\wedge p - (p\wedge q + q\wedge p) + \frac{1}{2}(q\wedge q + q'\wedge q') &(4.2.8b)\\
\mathrm{d}q &= -2q\wedge q &(4.2.8c)\\
\mathrm{d}q' &= -2(p\wedge q' + q'\wedge p) - (q\wedge q' + q'\wedge q) &(4.2.8d)
\end{aligned}$$

The equations of motion are obtained from the action [20]

$$S = -\frac{1}{2}\int \left(L^a \wedge *L^a + L^{a'} \wedge *L^{a'} - \bar{L}^1 \wedge L^2 - \bar{L}^2 \wedge L^1\right) \tag{4.2.9}$$

and can be written in terms of the lowercase 1-forms as [20]

$$\mathrm{d}{*p} = p \wedge *q + *q \wedge p + \frac{1}{2}(q \wedge q' + q' \wedge q) \tag{4.2.10a}$$

$$0 = p \wedge (*q - q') + (*q - q') \wedge p \tag{4.2.10b}$$

$$0 = p \wedge (q - *q') + (q - *q') \wedge p \tag{4.2.10c}$$

These equations are identical to the $AdS_5 \times S^5$ case discussed in [19]. From this we know that there are two one-parameter families of flat connections a given by

$$a = \alpha p + \beta{*p} + \gamma q + \delta q' \tag{4.2.11}$$

where

$$\alpha = -2\sinh^2\lambda \tag{4.2.12a}$$

$$\beta = \mp 2\sinh\lambda\cosh\lambda \tag{4.2.12b}$$

$$\gamma = 1 \pm \cosh\lambda \tag{4.2.12c}$$

$$\delta = \sinh\lambda \tag{4.2.12d}$$

4.3 Classical solutions

4.3.1 $AdS_3 \times S^1$ with $S^1 \subset S^3$

This solution can also be found in the $AdS_5 \times S^5$ sigma model [21] since it does not make use of the T^4 component of the considered space. It can be parametrized as

$$Y_0 = r_0 e^{i\kappa\tau}, \qquad Y_1 = r_1 e^{i\mathrm{w}\tau + ik\sigma}, \qquad X_1 = e^{iw\tau + im\sigma} \tag{4.3.1}$$

and $X_2 = Z_v = 0$. The constraint equations 3.3.1 are satisfied if $r_0^2 - r_1^2 = 1$. From the equations of motion 3.3.4 we get

$$\Lambda = m^2 - w^2, \Lambda' = -\kappa^2 = k^2 - \mathrm{w}^2 \tag{4.3.2}$$

Finally, the Virasoro constraints read

$$m^2 + w^2 - \kappa^2 r_0^2 + (k^2 + \mathrm{w}^2) r_1^2 = mw + k\mathrm{w} r_1^2 = 0 \tag{4.3.3}$$

The non-vanishing charges are

$$\mathcal{E} = r_0^2\kappa, \quad \mathcal{S} = r_1^2\mathrm{w}, \quad \mathcal{J}_1 = w \tag{4.3.4}$$

4.3.2 $AdS_3 \times S^1$ with $S^1 \subset T^4$

A simple variation of the previous solution can be obtained by choosing the S^1 to be part of T^4 instead of S^3. This solution is given by

$$Y_0 = r_0 e^{i\kappa\tau}, \qquad Y_1 = r_1 e^{i\mathrm{w}\tau + ik\sigma}, \qquad X_1 = 1, \qquad Z_1 = w\tau + m\sigma \tag{4.3.5}$$

and $X_2 = Z_2 = Z_3 = Z_4 = 0$. To satisfy the constraints, we require $r_0^2 - r_1^2 = 1$. The equations of motion imply

$$\Lambda = 0, \quad \Lambda' = -\kappa^2 = k^2 - \mathrm{w}^2 \tag{4.3.6}$$

The Virasoro constraints are identical to the previous case, given by

$$m^2 + w^2 - \kappa^2 r_0^2 + (k^2 + \mathrm{w}^2) r_1^2 = mw + k\mathrm{w} r_1^2 = 0 \tag{4.3.7}$$

The non-vanishing charges are

$$\mathcal{E} = r_0^2 \kappa, \quad \mathcal{S} = r_1^2 \mathrm{w}, \quad \mathcal{P}_1 = w \tag{4.3.8}$$

4.3.3 $\mathbb{R} \times S^3$

Another solution that is also used in the $AdS_5 \times S^5$ case [22] can be written as

$$Y_0 = e^{i\kappa\tau}, \qquad X_1 = \cos\gamma(\sigma) e^{iw_1\tau}, \qquad X_2 = \sin\gamma(\sigma) e^{iw_2\tau} \tag{4.3.9}$$

and $Y_1 = Z_v = 0$. Inserting this solution into the equations of motion leads to

$$0 = (w_1^2 + \Lambda - \gamma'^2)\cos\gamma - \gamma''\sin\gamma = (w_2^2 + \Lambda - \gamma'^2)\sin\gamma + \gamma''\cos\gamma = \kappa^2 + \tilde{\Lambda} \tag{4.3.10}$$

Solving the first to equations for Λ leads to a second order differential equation for $\gamma(\sigma)$, which can be written as

$$\gamma'' + \frac{1}{2}(w_2^2 - w_1^2)\sin 2\gamma = 0 \tag{4.3.11}$$

In the following, we will limit ourselves to the circular solution which satisfies

$$w_1 = w_2 = w, \quad \gamma = k\sigma \tag{4.3.12}$$

with an integer winding number k. This greatly simplifies the equations of motion and one obtains

$$\Lambda = k^2 - w^2, \quad \Lambda' = -\kappa^2 \tag{4.3.13}$$

The Virasoro constraints imply

$$\kappa^2 = w^2 + k^2 \tag{4.3.14}$$

The conserved charges are given by

$$\mathcal{E} = \kappa, \quad \mathcal{J}_1 = \mathcal{J}_2 = \frac{w}{2} \tag{4.3.15}$$

We see that the circular solution leads to equal spins $\mathcal{J}_1 = \mathcal{J}_2 = \mathcal{J}'$.

5. QUANTUM CORRECTIONS

We have seen that classically the $AdS_3 \times S^3$ string is very similar to the $AdS_5 \times S^5$ string theory, e.g. many classical solutions can be reproduced therein. However, as has been shown in the $AdS_5 \times S^5$ case, the quantum corrections obtain contributions from the full geometry (including fermions), so that it is clear that the quantum corrections to the same classical solutions will be different in the two backgrounds. We shall now compute the quantum corrections in $AdS_3 \times S^3 \times T^4$.

5.1 *General method*

5.1.1 Bosonic part

We start by expanding the bosonic fields around a classical solution $X_p \to X_p + \tilde{X}_p, Y_s \to Y_s + \tilde{Y}_s, Z_v \to Z_v + \tilde{Z}_v$ and expressing the fluctuation Lagrangian in terms of the fluctuation fields. The resulting Lagrangian is given by

$$\tilde{\mathcal{L}}_B = -\frac{1}{2} g^{\mu\nu} \left(\partial_\mu \tilde{X}_p^* \partial_\nu \tilde{X}_p + \partial_\mu \tilde{Y}_s^* \partial_\nu \tilde{Y}^s + \partial_\mu \tilde{Z}_v \partial_\nu \tilde{Z}_v \right) + \frac{1}{2} \left(\Lambda \tilde{X}_p^* \tilde{X}_p + \tilde{\Lambda} \tilde{Y}_s^* \tilde{Y}^s \right) \tag{5.1.1}$$

The fluctuation fields have to be restricted to the embedded $AdS_3 \times S^3 \times T^4$ space and thus satisfy the constraints that were imposed by including the Lagrange multipliers. These constraints are found to be

$$X_p^* \tilde{X}_p + \tilde{X}_p^* X_p = Y_s^* \tilde{Y}^s + \tilde{Y}_s^* Y^s = 0 \tag{5.1.2}$$

These constraints can be simplified by using a coordinate system that is moving along the classical solution. It can be written as

$$\begin{pmatrix} \tilde{X}_1 \\ \tilde{X}_2 \end{pmatrix} = \begin{pmatrix} e^{i\psi} & 0 \\ 0 & e^{i\phi} \end{pmatrix} \begin{pmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{pmatrix} \begin{pmatrix} g_1 + if_1 \\ g_2 + if_2 \end{pmatrix} \tag{5.1.3a}$$

$$\begin{pmatrix} \tilde{Y}_0 \\ \tilde{Y}_1 \end{pmatrix} = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{i\chi} \end{pmatrix} \begin{pmatrix} \cosh\rho & \sinh\rho \\ \sinh\rho & \cosh\rho \end{pmatrix} \begin{pmatrix} G_0 + iF_0 \\ G_1 + iF_1 \end{pmatrix} \tag{5.1.3b}$$

with f_p, g_p, F_s, G_s real. The constraints simplify to

$$g_1 = G_0 = 0 \tag{5.1.4}$$

The only remaining fields are $f_1, f_2, g_1, F_0, F_1, G_1, z_v = \tilde{Z}_v$. In conformal gauge, the Lagrangian can be brought in the form

$$L = \dot{x_p}^2 - x_p'^2 + K_{pq} x_p \dot{x_q} + W_{pq} x_p x_q' + M_{pq} x_p x_q \tag{5.1.5}$$

where $x_p, p = 1, \ldots, N$ are $N = 10$ independent fluctuation fields and K_{pq}, W_{pq}, M_{pq} are constant. Without loss of generality, we will assume that K and W are antisymmetric and M is symmetric, since the contribution from the antisymmetric part of M vanishes and the contributions from the symmetric parts of K and W are total derivatives. The equations of motion then read

$$0 = K_{pq}\dot{x_q} + W_{pq}x_q' + M_{pq}x_q - \ddot{x_p} + x_p'' \tag{5.1.6}$$

To solve these equations, assuming that the fields are periodic in σ, we use the ansatz

$$x_p(\tau, \sigma) = \sum_{n=-\infty}^{\infty} \sum_{I=1}^{2N} A_{p,I,n} e^{i(n\sigma + \omega_{I,n}\tau)} \tag{5.1.7}$$

Here, $I = 1, \ldots, 2N$ labels the different sets of solutions. Plugging this in we get

$$0 = \sum_{n=-\infty}^{\infty} \sum_{I=1}^{2N} \left(i\omega_{I,n}K_{pq} + inW_{pq} + M_{pq} + \left(\omega_{I,n}^2 - n^2\right)\delta_{pq}\right) A_{q,I,n} e^{i(n\sigma + \omega_{I,n}\tau)} \tag{5.1.8}$$

This set of linear equations has a non-vanishing solution for all σ, τ when there exists $A_{q,I,n}$ with

$$0 = \left(i\omega_{I,n}K_{pq} + inW_{pq} + M_{pq} + \left(\omega_{I,n}^2 - n^2\right)\delta_{pq}\right) A_{q,I,n} =: F_{pq}A_{q,I,n} \tag{5.1.9}$$

Such a solution only exists when F_{pq} has a zero eigenvalue and thus $\det F_{pq} = 0$.

5.1.2 Fermionic part

The fermionic frequencies can be obtained from the fermionic Lagrangian 3.2.4. The first step is the simplification of the Lagrangian by fixing the κ-symmetry. There are different choices for κ-symmetry fixing, of which the following two will be used in this thesis:

- $\theta^1 = \theta^2 = \theta$, which leads to

$$\mathcal{L}_F^{(2)} = \bar{\theta}\left(2i\rho^\mu\left(\partial_\mu + \frac{1}{4}\omega_\mu^{MN}\Gamma_{MN}\right) + \frac{i}{4}\epsilon^{\mu\nu}\rho_\mu e_\nu^A \Gamma_0\Gamma_1\Gamma_2\Gamma_A\right)\theta \tag{5.1.10}$$

- $\theta_-^I = 0$, where

$$\theta_-^I = \mathcal{P}_-^{IJ}\theta^J = \frac{1}{2}(\delta^{IJ} - i\epsilon^{IJ}\Gamma^0\Gamma^1)\theta^J \tag{5.1.11}$$

  This condition can be satisfied by defining

$$\theta = \theta^1 = i\Gamma^0\Gamma^1\theta^2 \tag{5.1.12}$$

  The Lagrangian then reads

$$\mathcal{L}_F^{(2)} = \bar{\theta}\left(i\eta^{\mu\nu}(\Delta_{\mu\nu} - \tilde{\Delta}_{\mu\nu}) - i\epsilon^{\mu\nu}(\Delta_{\mu\nu} + \tilde{\Delta}_{\mu\nu})\right)\theta \tag{5.1.13}$$

  with

$$\Delta_{\mu\nu} = \rho_\mu\left(\partial_\mu + \frac{1}{4}\omega_\mu^{MN}\Gamma_{MN} - \frac{i}{2}e_\nu^A\Gamma_0\Gamma_1\Gamma_2\Gamma_A\Gamma^0\Gamma^1\right) \tag{5.1.14}$$

  and

$$\tilde{\Delta}_{\mu\nu} = \Gamma^{1\dagger}\Gamma^{0\dagger}\Delta_{\mu\nu}\Gamma^0\Gamma^1 \tag{5.1.15}$$

In both cases the Lagrangian has the form

$$\mathcal{L}_F^{(2)} = \bar{\theta} D_F \theta \tag{5.1.16}$$

This directly leads to the very simple equations of motion

$$D_F \theta = 0 \tag{5.1.17}$$

As for the bosonic frequencies, one can use an ansatz that is periodic in σ and plug it into the equations of motion. Let

$$\theta(\tau, \sigma) = \sum_{n=-\infty}^{\infty} \sum_{I=1}^{2N} \vartheta_{I,n} e^{i(n\sigma + \omega_{I,n}\tau)} \tag{5.1.18}$$

with constant spinors $\vartheta_{I,n}$. The equations of motion translate into

$$0 = \sum_{n=-\infty}^{\infty} \sum_{I=1}^{2N} \hat{D}_F \vartheta_{I,n} e^{i(n\sigma + \omega_{I,n}\tau)} \tag{5.1.19}$$

where $\hat{D}_F$ can be obtained from D_F by the substitutions $\partial_\tau \to i\omega$, $\partial_\sigma \to in$. They are satisfied for all σ, τ when

$$0 = \hat{D}_F \vartheta_{I,n} \tag{5.1.20}$$

The fermionic frequencies are the values of ω for which $\det \hat{D}_F = 0$. Before computing the determinant, it is useful to choose a basis in which $\hat{D}_F$ has block diagonal form. This can be done by choosing combinations of gamma matrices that commute with $\hat{D}_F$, i.e. $\tilde{\Gamma} = \Gamma_0^{\epsilon_0} \Gamma_1^{\epsilon_1} \dots \Gamma_9^{\epsilon_9}$ with $\tilde{\Gamma} \hat{D}_F = \hat{D}_F \tilde{\Gamma}$. Diagonalization of $\tilde{\Gamma}$ is much simpler than a direct computation and leads to a basis in which $\hat{D}_F$ has block diagonal form.

5.2 $AdS_3 \times S^1$ with $S^1 \subset S^3$

5.2.1 Bosonic part

The quadratic fluctuations are computed by inserting for each of the fields a small deviation $X_p \to X_p + \delta X_p$, $Y_s \to Y_s + \delta Y_s$ and deriving the constraints on the fields δX_p, δY_s that follow from the classical equations of motion [21].

For the transverse bosonic fluctuations in the S^3 the Lagrangian is

$$\delta\mathcal{L} = -\frac{1}{2} \partial_a \delta X_p \partial^a \delta X_p^* + \frac{1}{2} \Lambda_2 \delta X_p \delta X_p^* \tag{5.2.1}$$

For the present case only X_2 decouples from the other fields and thus we obtain two real bosons of mass-squared $\nu^2 = \Lambda_2 = \mathcal{J}^2 - m^2$.

$$\omega_n^S = \frac{1}{2\kappa} \sqrt{n^2 + \nu^2} \tag{5.2.2}$$

Likewise the torus-directions are determined trivially, and due to the absence of the Lagrange multiplier yield four real massless bosons

$$\omega_n^T = 4 \times \frac{|n|}{2\kappa} \tag{5.2.3}$$

The fluctuations in the internal bosonic directions (i.e., along the $AdS_3 \times S^1$) can be computed the same way as in [21] by a method explained in 5.1.1. Starting from the Lagrangian 5.1.1 and setting

$$\delta X_1 = e^{iw\tau + im\sigma}(g_1 + if_1), \quad \begin{pmatrix} \delta Y_0 \\ \delta Y_1 \end{pmatrix} = \begin{pmatrix} e^{i\kappa\tau} & 0 \\ 0 & e^{i\mathrm{w}\tau + ik\sigma} \end{pmatrix} \begin{pmatrix} r_0 & r_1 \\ r_1 & r_0 \end{pmatrix} \begin{pmatrix} G_0 + iF_0 \\ G_1 + iF_1 \end{pmatrix} \tag{5.2.4}$$

$$\delta X_1 = e^{iw\tau + im\sigma}(g_1 + if_1), \quad \delta Y_0 = e^{i\kappa\tau}(G_0 + iF_0), \quad \delta Y_1 = e^{i\mathrm{w}\tau + ik\sigma}(G_1 + iF_1) \tag{5.2.5}$$

and solving the constraints 3.3.1 we find the final form for the Lagrangian

$$\begin{aligned} \delta L = &\frac{1}{2}\left((w^2 - m^2)f_1^2 - \kappa^2 F_0^2 + (\mathrm{w}^2 - k^2)(F_1^2 + G_1^2) - \frac{r_1}{r_0}\kappa^2 G_1^2\right) \\ &+ \frac{1}{2}\left(\dot{f_1}^2 - f_1'^2 - \dot{F_0}^2 + F_0'^2 + \dot{F_1}^2 - F_1'^2 + \frac{1}{r_0^2}\dot{G_1}^2 - \frac{1}{r_0^2}G_1'^2\right) \\ &+ \kappa\frac{r_1}{r_0}F_0\dot{G}_1 + F_1(kG_1' - \mathrm{w}\dot{G}_1) - G_1\left(\kappa\frac{r_1}{r_0}\dot{F}_0 - \mathrm{w}\dot{F}_1 + kF_1'\right) \end{aligned} \tag{5.2.6}$$

Since $\kappa^2 = \mathrm{w}^2 - k^2$ and $\nu^2 = w^2 - m^2$, the first line can be written as

$$\frac{1}{2}\nu^2(g_1^2 + f_1^2) + \frac{1}{2}\kappa^2(-F_0^2 - G_0^2 + F_1^2 + G_1^2) \equiv 0 \tag{5.2.7}$$

After rescaling $F_0 \to iF_0, G_1 \to r_0 G_1$ we follow the procedure stated in the appendix. The bosonic frequencies are the solutions to the equation

$$0 = (n-\omega)^2(n+\omega)^2\left((\omega^2 - n^2)^2 + 4r_1^2\kappa^2\omega^2 - 4(1 + r_1^2)\left(\sqrt{\kappa^2 + k^2}\omega - kn\right)^2\right) \tag{5.2.8}$$

The first two factors correspond to massless modes which are canceled by the conformal ghost contributions.

5.2.2 Fermionic part

We start by computing the vielbein, which can be written as

$$E_m^M = \mathrm{diag}(\cosh\rho, 1, \sinh\rho, 1, 1, 0, 1, 1, 1, 1) \tag{5.2.9}$$

One of the components vanishes due to a metric singularity at $\gamma = 0$. The non-vanishing components of the Lorentz connection are (up to symmetry)

$$\omega_0^{01} = \sinh\rho, \quad \omega_2^{12} = -\cosh\rho, \quad \omega_5^{35} = -1 \tag{5.2.10}$$

After projecting onto the classical solution, we find for the vielbein

$$e_\tau^0 = \kappa\cosh\rho, \quad e_\tau^2 = \mathrm{w}\sinh\tau, \quad e_\tau^4 = w, \quad e_\sigma^2 = k\sinh\rho, \quad e_\sigma^4 = m \tag{5.2.11}$$

and the Lorentz connection

$$\omega_\tau^{01} = \kappa\sinh\rho, \quad \omega_\tau^{12} = -\mathrm{w}\cosh\rho, \quad \omega_\sigma^{12} = -k\cosh\rho \tag{5.2.12}$$

A further computation of the fermionic frequencies has been attempted, but failed due to the large complexity of the resulting expression for $\det D_F$. An extensive use of computer algebra systems and simplifications did not solve the problem. More work and further simplifications are needed, but would exceed the completion time of this thesis.

5.3 $AdS_3 \times S^1$ with $S^1 \subset T^4$

5.3.1 Bosonic part

From the S^3 directions, we get 3 free bosons with mass squared $\nu^2 = \Lambda_2 = \mathcal{J}^2 - m^2$,

$$\omega_n^S = \frac{1}{2\kappa}\sqrt{n^2+\nu^2} \tag{5.3.1}$$

The torus directions also decouple and contribute 4 massless bosons

$$\omega_n^T = 4 \times \frac{|n|}{2\kappa} \tag{5.3.2}$$

For the internal AdS_3 directions we use the same computation as in the previous case. Setting

$$\begin{pmatrix}\delta Y_0\\ \delta Y_1\end{pmatrix} = \begin{pmatrix}e^{i\kappa\tau} & 0\\ 0 & e^{i\mathrm{w}\tau+ik\sigma}\end{pmatrix}\begin{pmatrix}r_0 & r_1\\ r_1 & r_0\end{pmatrix}\begin{pmatrix}G_0+iF_0\\ G_1+iF_1\end{pmatrix} \tag{5.3.3}$$

and solving the constraints 3.3.1 we find the final form for the Lagrangian

$$\begin{aligned}\delta L =& \frac{1}{2}\left(-\kappa^2F_0^2 + (\mathrm{w}^2-k^2)(F_1^2+G_1^2) - \frac{r_1}{r_0}\kappa^2 G_1^2\right)\\ &+\frac{1}{2}\left(-\dot{F_0}^2 + F_0'^2 + \dot{F_1}^2 - F_1'^2 + \frac{1}{r_0^2}\dot{G_1}^2 - \frac{1}{r_0^2}G_1'^2\right)\\ &+\kappa\frac{r_1}{r_0}F_0\dot{G}_1 + F_1(kG_1' - \mathrm{w}\dot{G}_1) - G_1\left(\kappa\frac{r_1}{r_0}\dot{F}_0 - \mathrm{w}\dot{F}_1 + kF_1'\right)\end{aligned} \tag{5.3.4}$$

Similarly to the $S^1 \subset S^3$ case, the first line can be written as

$$\frac{1}{2}\kappa^2(-F_0^2 - G_0^2 + F_1^2 + G_1^2) \equiv 0 \tag{5.3.5}$$

After rescaling $F_0 \to iF_0, G_1 \to r_0 G_1$ we follow the procedure stated at the beginning of this chapter. The bosonic frequencies are the solutions to the equation

$$0 = (n-\omega)(n+\omega)\left((\omega^2-n^2)^2 + 4r_1^2\kappa^2\omega^2 - 4(1+r_1^2)\left(\sqrt{\kappa^2+k^2}\omega - kn\right)^2\right) \tag{5.3.6}$$

The massless modes originating from the first two factors are again canceled by the conformal ghosts.

5.3.2 Fermionic part

We start by computing the vielbein, which can be written as

$$E_m^M = \mathrm{diag}(\cosh\rho, 1, \sinh\rho, 1, 1, 0, 1, 1, 1, 1) \tag{5.3.7}$$

One of the components vanishes due to a metric singularity at $\gamma = 0$. The non-vanishing components of the Lorentz connection are (up to symmetry)

$$\omega_0^{01} = \sinh\rho, \quad \omega_2^{12} = -\cosh\rho, \quad \omega_5^{35} = -1 \tag{5.3.8}$$

After projecting onto the classical solution, we find for the vielbein

$$e^0_\tau = \kappa\cosh\rho, \quad e^2_\tau = \mathrm{w}\sinh\tau, \quad e^6_\tau = w, \quad e^2_\sigma = k\sinh\rho, \quad e^6_\sigma = m \tag{5.3.9}$$

and the Lorentz connection

$$\omega^{01}_\tau = \kappa\sinh\rho, \quad \omega^{12}_\tau = -\mathrm{w}\cosh\rho, \quad \omega^{12}_\sigma = -k\cosh\rho \tag{5.3.10}$$

As in the $S^1 \subset \S^3$ case, a computation of the fermionic frequencies has been attempted without success. The expression obtained for $\det D_F$ is slightly simpler than in the previous case, but still too complex to solve it before this thesis has to be finished.

5.4 $\mathbb{R}\times S^3$

5.4.1 Bosonic part

Once again, we start from the fluctuation Lagrangian obtained by replacing $X_p \to X_p + \delta X_p$. The torus directions contribute 4 massless bosons

$$\omega^T_n = 4 \times \frac{|n|}{2\kappa} \tag{5.4.1}$$

Y_1 decouples and yields 2 massive bosons with mass squared $\kappa^2 = \Lambda_1$:

$$\omega^S_n = \frac{1}{2\kappa}\sqrt{n^2+\kappa^2} \tag{5.4.2}$$

For the internal bosonic directions, we set

$$\begin{pmatrix}\delta X_1\\ \delta X_2\end{pmatrix} = e^{iw\tau}\begin{pmatrix}\cos k\sigma & -\sin k\sigma\\ \sin k\sigma & \cos k\sigma\end{pmatrix}\begin{pmatrix}g_1+if_1\\ g_2+if_2\end{pmatrix}, \quad \delta Y_0 = e^{i\kappa\tau}(G_0+iF_0) \tag{5.4.3}$$

The constraints imply

$$g_1 = G_0 = 0 \tag{5.4.4}$$

Plugging this in, we end up with the Lagrangian

$$\begin{aligned}\delta L = &\frac{1}{2}\left((w^2-k^2)(f_1^2+f_2^2+g_2^2) - \kappa^2 F_0^2\right)\\ &+\frac{1}{2}\left(\dot{f_1}^2 - f_1'^2 + \dot{f_2}^2 - f_2'^2 + \dot{g_2}^2 - g_2'^2 - \dot{F_0}^2 + F_0'^2\right)\\ &+ f_2(kf_1' - w\dot{g}_2) - kf_1f_2' + wg_2\dot{f}_2\end{aligned} \tag{5.4.5}$$

The first line can be written as

$$\frac{1}{2}\kappa^2(f_1^2+g_1^2+f_2^2+g_2^2-F_0^2-G_0^2) \equiv 0 \tag{5.4.6}$$

After rescaling $F_0 \to iF_0$ we can apply the procedure from 5.1.1 and find the bosonic frequencies as solutions of the equation

$$0 = (n-\omega)^2(n+\omega)^2\left(\omega^4 - 2(n^2+2w^2)\omega^2 + n^4 - 4k^2n^2\right) \tag{5.4.7}$$

As in the previous cases, the contributions from the first two factors are canceled by the conformal ghost contributions.

5.4.2 Fermionic part

We start by computing the vielbein, which can be written as

$$E_m^M = \text{diag}(1,1,0,1,\cos(k\sigma),\sin(k\sigma),1,1,1,1) \tag{5.4.8}$$

The vanishing component is due to a metric singularity at $\rho = 0$ (the center of AdS_3), but it will not have any impact on the following computation. The non-vanishing components of the Lorentz connection are (up to symmetry)

$$\omega_2^{12} = -1, \quad \omega_4^{34} = \sin(k\sigma), \quad \omega_5^{35} = -\cos(k\sigma) \tag{5.4.9}$$

After projecting onto the classical solution, we find for the vielbein

$$e_\tau^0 = \kappa, \quad e_\tau^4 = w\cos(k\sigma), \quad e_\tau^5 = w\sin(k\sigma), \quad e_\sigma^3 = k \tag{5.4.10}$$

and the Lorentz connection

$$\omega_\tau^{34} = w\sin(k\sigma), \quad \omega_\tau^{35} = -w\cos(k\sigma) \tag{5.4.11}$$

The operator D_F computed using these coefficients is given by

$$\begin{aligned} D_F = i\kappa\omega\Gamma_0 &- \frac{ikw}{2}\cos(k\sigma)\Gamma_{01234} - \frac{ikw}{2}\sin(k\sigma)\Gamma_{01235} + \frac{w\kappa}{2}\sin(k\sigma)\Gamma_{034} - \frac{w\kappa}{2}\cos(k\sigma)\Gamma_{035} \\ &+ \frac{ik\kappa}{2}\Gamma_{123} - ikn\Gamma_3 + \frac{w^2}{2}\Gamma_{345} + iw\omega\cos(k\sigma)\Gamma_4 + iw\omega\sin(k\sigma)\Gamma_5 \end{aligned} \tag{5.4.12}$$

The σ dependence can be eliminated by performing a rotation in the 45-plane. Replacing

$$D_F \to S^{-1}D_F S, \quad S = \exp\left(-\frac{1}{2}k\sigma\Gamma_{45}\right) \tag{5.4.13}$$

we end up with

$$D_F = i\kappa\omega\Gamma_0 - \frac{ikw}{2}\Gamma_{01234} - \frac{w\kappa}{2}\Gamma_{035} + \frac{ik\kappa}{2}\Gamma_{123} - ikn\Gamma_3 + \frac{w^2}{2}\Gamma_{345} + iw\omega\Gamma_4 \tag{5.4.14}$$

A possible choice for a maximal subset of mutually commuting combinations of gamma matrices that commute with D_F is given by $\Gamma_{12}, \Gamma_{67}, \Gamma_{89}$. Since the gamma matrices $\Gamma_6, \Gamma_7, \Gamma_8, \Gamma_9$ do not contribute to D_F, we can restrict ourselves to a representation of the first 6 gamma matrices. Then the determinant of D_F is found to be

$$\det D_F = \frac{k^8}{256}\left(\kappa^4 - 4\left(k^2 - 6n^2 + 2\omega^2\right)\kappa^2 + 4\left(k^2 - 2n^2 + 2\omega^2\right)^2\right)^2 \tag{5.4.15}$$

The fermionic frequencies are the roots of this determinant. We see that

$$\omega = \pm\frac{1}{2}\sqrt{-2k^2 + 4n^2 + \kappa^2 \pm 4in\kappa} \tag{5.4.16}$$

In addition, there are four free fermions from the T^4 component of the superspace. Thus the sum of the fermionic frequencies (weighted with appropriate signs) is given by

$$\begin{aligned} \omega_n^F &= -4n - 2\left(\sqrt{-2k^2 + 4n^2 + \kappa^2 - 4in\kappa} + \sqrt{-2k^2 + 4n^2 + \kappa^2 + 4in\kappa}\right) \\ &= -4n - 2\left(\sqrt{\mathcal{J}^2 - k^2 + 4n\left(n - i\sqrt{\mathcal{J}^2 + k^2}\right)} + \sqrt{\mathcal{J}^2 - k^2 + 4n\left(n + i\sqrt{\mathcal{J}^2 + k^2}\right)}\right) \end{aligned} \tag{5.4.17}$$

5.4.3 Large $1/\mathcal{J}$ expansion

It is now straight forward to obtain the λ' expansion of the one-loop energy shift

$$\delta E = \delta E_0 + 2\sum_{n=1}^{\infty} \omega_n^S + \omega_n^T + \omega_n^{AdS} + \omega_n^F \,. \tag{5.4.18}$$

- Analytic terms $O(\frac{1}{\mathcal{J}^{2n}})$ (via zeta-function regularization): first expand in $\frac{1}{\mathcal{J}}$, and then zeta-function regularize the sums at each order. See [23].
- Non-analytic terms $O(\frac{1}{\mathcal{J}^{2n+1}})$: integral approximation. See [24, 25, 26].
- Exponential terms $e^{\frac{1}{\mathcal{J}}}$ [26]

Given the above results for the fluctuation frequencies at one-loop we obtain

$$\begin{aligned}
\delta E &= \delta E_0 + \sum_{n=1}^{\infty} \omega_n^{\text{bosonic}} + \omega_n^F \\
&= \sum_{n=1}^{\infty} \Bigg(2\sqrt{\mathcal{J}^2 + k^2 + n^2} \\
&\quad + \sqrt{2\mathcal{J}^2 + n^2 - 2\sqrt{\mathcal{J}^4 + (\mathcal{J}^2 + k^2)\, n^2}} + \sqrt{2\mathcal{J}^2 + n^2 + 2\sqrt{\mathcal{J}^4 + (\mathcal{J}^2 + k^2)\, n^2}} \\
&\quad - 2\sqrt{\mathcal{J}^2 - k^2 + 4n\left(n - i\sqrt{\mathcal{J}^2 + k^2}\right)} - 2\sqrt{\mathcal{J}^2 - k^2 + 4n\left(n + i\sqrt{\mathcal{J}^2 + k^2}\right)}\Bigg)
\end{aligned} \tag{5.4.19}$$

This sum in particular converges as the summand is order $O(\frac{1}{n^2})$ for large n.

We wish to evaluate this in the large $\mathcal{J}$ expansion, in particular, in order to compare to the $AdS_5 \times S^5$ case and also possibly to a quantum string Bethe ansatz. We shall present the evaluation by computing the analytic terms via zeta-function regularization.

5.4.4 Analytic Terms

The analytic terms which come in a power series expansion in $\frac{1}{\mathcal{J}^2}$ are obtained by expanding the summand naively in $\frac{1}{\mathcal{J}}$ and then summing at each order in $\frac{1}{\mathcal{J}}$. We get for the non-zero modes

$$\begin{aligned}
\delta E|_{\text{non-zeromodes}} &= \sum_n \frac{1}{2}\left(6k^2 - 29n^2 + n\sqrt{n^2 - 4k^2}\right)\frac{1}{\mathcal{J}^2} \\
&\quad + \sum_n \frac{1}{8}\left(2k^4 - 194n^2k^2 + 765n^4 - n^3\sqrt{n^2 - 4k^2}\right)\frac{1}{\mathcal{J}^4} \\
&\quad + \sum_n \frac{1}{16}\left(6k^6 - 474n^2k^4 + 6403n^4k^2 - 18429n^6 + n^3\sqrt{n^2 - 4k^2}\left(n^2 - k^2\right)\right)\frac{1}{\mathcal{J}^6} \\
&\quad + O\left(\frac{1}{\mathcal{J}^8}\right)
\end{aligned} \tag{5.4.20}$$

As in the $AdS_5 \times S^5$ case, the partial sums at each order diverge and we can evaluate them formally by ζ-function evaluation. Let us denote the energies at each order in $\frac{1}{\mathcal{J}^p}$ by $E^{(p)}$.

At order $\frac{1}{\mathcal{J}^2}$ the sums have asymptotics $-14n^2 + 2k^2 + O(\frac{1}{n^2})$. Zeta function regularization means that the following definition of the sums via the Riemann ζ-function are used

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{5.4.21}$$

Furthermore

$$\zeta(-m) = -\frac{B_{m+1}}{m+1}\,, \qquad m \in \mathbb{N}\,. \tag{5.4.22}$$

In particular, $\zeta(0) = -\frac{1}{2}$ and for all even m the terms vanish.

Thus by zeta-function regularization at order $\frac{1}{\mathcal{J}^2}$ we need to subtract $-14n^2 + 2k^2$ and add $-\frac{1}{2}(2k^2)$. Then

$$\delta E^{(2)} = \frac{1}{2}\left(n\sqrt{n^2 - 4k^2} - n^2\right)\,. \tag{5.4.23}$$

Likewise

$$\delta E^{(4)} = \frac{1}{8}\left(-4k^4 - 2n^2k^2 + n^3\left(n - \sqrt{n^2 - 4k^2}\right)\right)\,, \tag{5.4.24}$$

and

$$\delta E^{(6)} = \frac{1}{16}\left(3k^2n^4 - n^6 + n^3\sqrt{n^2 - 4k^2}\left(n^2 - k^2\right)\right)\,. \tag{5.4.25}$$

5.4.5 Discussion

We see that the quantum corrections that have been obtained in our computation have a form which is similar to the quantum corrections in the $AdS_5 \times S^5$ case. [26] The difference in the analytic terms reflects the fact that the quantum corrections are influenced not only by the classical string trajectory, but receive contributions from the whole supergeometry of the string background. More work is required to compute the non-analytic and exponential terms.

6. CONCLUSION

In this work we have shown that the $AdS_3 \times S^3 \times T^4$ superstring with RR flux is a classically integrable model. We have shown that an infinite set of conserved charges exists and computed the conserved charges in different bases of the symmetry superalgebra.

We have furthermore constructed several spinning string solutions and computed their energies at the classical level and quantum corrections at one-loop order. We have seen that the divergent terms in the bosonic and fermionic contributions to the quantum corrections cancel, as one would expect from a supersymmetric model. We have then computed the analytic terms of the $\frac{1}{\mathcal{J}}$ expansion. From this derivation we have seen that the results are have a similar structure as those obtained for spinning strings in $AdS_5 \times S^5$.

During this work, an extensive use of Mathematica has shown to be useful. Two Mathematica packages have been developed for algebraic simplifications and applied to the computations done in this work. The Grassmann package helps in dealing with Grassmann algebras and in constructing representations of superalgebras. The Clifford package performs various simplifications in Clifford algebras and utilizes a simple, but fast algorithm to compute products and inverses of Clifford algebra elements.

There are various possibilities for further studies:

- As in the $AdS_5 \times S^5$ case, the spectrum may be computed using a string Bethe ansatz and compared to the result obtained by explicit one-loop calculations. [23, 27] This gives us the possibility to test whether the Bethe ansatz is valid for the computation of string spectra and may give some hints on how the different backgrounds influence the structure of the quantum corrections.
- The origin of analytic, non-analytic and exponential terms in the $\frac{1}{\mathcal{J}}$ expansion may be analyzed in more detail. [26]
- The effects of zeta-function regularization on the evaluation of quantum corrections to spinning strings may be discussed as it has been done in the case of $AdS_5 \times S^5$. [25]
- Exact expressions for the quantum corrections may be computed as in [26].
- The $AdS_3 \times S^1$ solutions may be examined in more detail, as it has already be done for the $\mathbb{R} \times S^3$ solution.

APPENDIX

A. CONVENTIONS AND NOTATION

A.1 Coordinate indices

We use the following convention for the indices:

$$
\begin{array}{rl}
a, b, \ldots = 1,2,3 & AdS_3 \text{ tangent space indices} \\
a', b', \ldots = 1', 2', 3' & S^3 \text{ tangent space indices} \\
\tilde{a}, \tilde{b}, \ldots = \tilde{1}, \tilde{2}, \tilde{3}, \tilde{4} & T^4 \text{ tangent space indices} \\
\hat{a}, \hat{b}, \ldots = 1,2,3,1',2',3' & AdS_3 \times S^3 \text{ tangent space indices} \\
A, A', \tilde{A}, \hat{A} & \text{same as above for labeling vielbeins} \\
m, n, \ldots = 1,2,3,1',2',3',\tilde{1},\tilde{2},\tilde{3},\tilde{4} & AdS_3 \times S^3 \times T^4 \text{ tangent space indices} \\
M, N, \ldots & AdS_3 \times S^3 \times T^4 \text{ vielbein indices} \\
P, Q, \ldots = 0,1,2,3 & \mathbb{R}^{1,3} \supset AdS_3 \text{ coordinate indices} \\
p, q, \ldots = 0,1 & \mathbb{R}^{1,3} \supset AdS_3 \text{ coordinate indices (complexified)} \\
S, T, \ldots = 1,2,3,4 & \mathbb{R}^4 \supset S^3 \text{ coordinate indices} \\
s, t, \ldots = 1,2 & \mathbb{R}^4 \supset S^3 \text{ coordinate indices (complexified)} \\
v, w, \ldots = 1,2,3,4 & T^4 \text{ coordinate indices} \\
i, j, \ldots = 0,1,2,3 & \mathrm{SO}(2,2) \text{ vector indices} \\
i', j', \ldots = 0',1',2',3' & \mathrm{SO}(4) \text{ vector indices} \\
I, J, \ldots = 1,2 & \text{labels the two sets of spinors} \\
\mu, \nu, \ldots = \tau, \sigma & \text{world sheet coordinates}
\end{array}
$$

A.2 Convention for gamma matrices

The gamma matrices of the 6-dimensional Clifford algebra used in this thesis can be written as

$$\gamma^0 = i\sigma^3 \otimes \mathbb{1}_2 \otimes \sigma^1 \qquad \gamma^1 = \sigma^1 \otimes \mathbb{1}_2 \otimes \sigma^1 \qquad \gamma^2 = \sigma^2 \otimes \mathbb{1}_2 \otimes \sigma^1 \tag{A.2.1a}$$

$$\gamma^3 = \mathbb{1}_2 \otimes \sigma^1 \otimes \sigma^2 \qquad \gamma^4 = \mathbb{1}_2 \otimes \sigma^2 \otimes \sigma^2 \qquad \gamma^5 = \mathbb{1}_2 \otimes \sigma^3 \otimes \sigma^2 \tag{A.2.1b}$$

From these, the 10-dimensional gamma matrices are obtained by defining $\Gamma^A = \gamma^A \otimes \mathbb{1}_4$, $\Gamma^{A'} = \gamma^{A'} \otimes \mathbb{1}_4$ and

$$\Gamma^6 = \mathbb{1}_4 \times \sigma_3 \times \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix} \qquad \Gamma^7 = \mathbb{1}_4 \times \sigma_3 \times \begin{pmatrix} 0 & i\sigma^1 \\ -i\sigma^1 & 0 \end{pmatrix} \tag{A.2.2}$$

$$\Gamma^8 = \mathbb{1}_4 \times \sigma_3 \times \begin{pmatrix} 0 & i\sigma^2 \\ -i\sigma^2 & 0 \end{pmatrix} \qquad \Gamma^9 = \mathbb{1}_4 \times \sigma_3 \times \begin{pmatrix} 0 & i\sigma^3 \\ -i\sigma^3 & 0 \end{pmatrix} \tag{A.2.3}$$

B. BASES OF $PSU(1,1|2) \times \widetilde{PSU}(1,1|2)$

B.1 The canonical basis

The most obvious way to construct the symmetry algebra is to start with the purely bosonic symmetry algebra of $AdS_3 \times S^3$, which is $\mathfrak{so}(2,2) \times \mathfrak{so}(4)$. The generators of this algebra are

$$P_a, J_{ab}, P_{a'}, J_{a'b'} \tag{B.1.1}$$

where $a, b = 1, 2, 3$ and $a', b' = 1', 2', 3'$. Their commutation relations can be put in a simple form by defining

$$M_{0a} = P_a, \qquad M_{ab} = J_{ab}, \qquad M_{0a'} = P_{a'}, \qquad M_{a'b'} = J_{a'b'} \tag{B.1.2}$$

This allows us to write

$$[M_{ij}, M_{kl}] = \eta_{jk} M_{il} + \eta_{il} M_{jk} - \eta_{ik} M_{jl} - \eta_{jl} M_{ik} \tag{B.1.3a}$$

$$[M_{i'j'}, M_{k'l'}] = \delta_{j'k'} M_{i'l'} + \delta_{i'l'} M_{j'k'} - \delta_{i'k'} M_{j'l'} - \delta_{j'l'} M_{i'k'} \tag{B.1.3b}$$

where $\eta = (- + + -)$. To incorporate the superspace coordinates into this algebra, we need to add 16 supersymmetry generators $Q_{I\alpha\alpha'}$ and $\bar{Q}_{I\alpha\alpha'}$, $I = 1, 2$, $\alpha = 1, 2$, $\alpha' = 1, 2$. The remaining (anti)commutation relations are

$$[P_a, Q_I] = -\frac{i}{2}\epsilon_{IJ}\gamma_a Q_J \qquad [P_{a'}, Q_I] = \frac{1}{2}\epsilon_{IJ}\gamma_{a'} Q_J \tag{B.1.4a}$$

$$[J_{ab}, Q_I] = -\frac{1}{2}\gamma_{ab} Q_I \qquad [J_{a'b'}, Q_I] = -\frac{1}{2}\gamma_{a'b'} Q_I \tag{B.1.4b}$$

$$\left[P_a, \bar{Q}_I\right] = \frac{i}{2}\bar{Q}_J \epsilon_{JI}\gamma_a \qquad \left[P_{a'}, \bar{Q}_I\right] = -\frac{1}{2}\bar{Q}_J \epsilon_{JI}\gamma_{a'} \tag{B.1.4c}$$

$$\left[J_{ab}, \bar{Q}_I\right] = \frac{1}{2}\bar{Q}_I \gamma_{ab} \qquad \left[J_{a'b'}, \bar{Q}_I\right] = \frac{1}{2}\bar{Q}_I \gamma_{a'b'} \tag{B.1.4d}$$

and finally

$$\left\{Q_I, \bar{Q}_J\right\} = 2\delta_{IJ}\left(iP_a\gamma^a - P_{a'}\gamma^{a'}\right) + \epsilon_{IJ}\left(J_{ab}\gamma^{ab} - J_{a'b'}\gamma^{a'b'}\right) \tag{B.1.4e}$$

The gamma matrices are defined as

$$\gamma^1 = i\sigma^3, \gamma^2 = \sigma^1, \gamma^3 = \sigma^2, \gamma^{1'} = \sigma^1, \gamma^{2'} = \sigma^2, \gamma^{3'} = \sigma^3 \tag{B.1.5}$$

We see that $\mathcal{G}$ respects a $\mathbb{Z}_4$ grading, i.e.

$$\mathcal{G} = \mathcal{G}_0 + \mathcal{G}_1 + \mathcal{G}_2 + \mathcal{G}_3 \tag{B.1.6}$$

and

$$[\mathcal{G}_m, \mathcal{G}_n\} \subseteq \mathcal{G}_{m+n} \tag{B.1.7}$$

for $m, n \in \mathbb{Z}_4$. Here $[.,.\}$ denotes the anticommutator between two fermionic operators and the commutator otherwise. We find that

$$J_{a'b'}, J_{ab} \in \mathcal{G}_0, \quad Q_{1\alpha\alpha'}, \bar{Q}_{1\alpha\alpha'} \in \mathcal{G}_1, \quad P_a, P_{a'} \in \mathcal{G}_2, \quad Q_{2\alpha\alpha'}, \bar{Q}_{2\alpha\alpha'} \in \mathcal{G}_3 \tag{B.1.8}$$

B.2 The covariant basis

Since $\mathfrak{so}(2,2) \cong \mathfrak{su}(1,1) \times \mathfrak{su}(1,1)$ and $\mathfrak{so}(4) \cong \mathfrak{su}(2) \times \mathfrak{su}(2)$, it is possible to choose a basis in which the factor algebras are evident. Let $m^{\alpha}{}_{\beta}, \tilde{m}^{\alpha}{}_{\beta}, \alpha, \beta = 1,2$ and $m^{\alpha'}{}_{\beta'}, \tilde{m}^{\alpha'}{}_{\beta'}, \alpha', \beta' = 1,2$ denote the two sets of $\mathfrak{su}(1,1)$ and $\mathfrak{su}(2)$ generators, respectively. They satisfy the commutation relations

$$\left[\overset{(\sim)}{m}{}^{\alpha}{}_{\beta}, \overset{(\sim)}{m}{}^{\gamma}{}_{\delta}\right] = \delta^{\gamma}_{\beta}\overset{(\sim)}{m}{}^{\alpha}{}_{\delta} - \delta^{\alpha}_{\delta}\overset{(\sim)}{m}{}^{\gamma}{}_{\beta} \qquad \left[\overset{(\sim)}{m}{}^{\alpha'}{}_{\beta'}, \overset{(\sim)}{m}{}^{\gamma'}{}_{\delta'}\right] = \delta^{\gamma'}_{\beta'}\overset{(\sim)}{m}{}^{\alpha'}{}_{\delta'} - \delta^{\alpha'}_{\delta'}\overset{(\sim)}{m}{}^{\gamma'}{}_{\beta'} \tag{B.2.1}$$

The 16 supercharges are denoted $q^{\alpha}_{\alpha'}, q^{\alpha'}_{\alpha}, \tilde{q}^{\alpha}_{\alpha'}, \tilde{q}^{\alpha'}_{\alpha}$ and satisfy

$$\left[\overset{(\sim)}{m}{}^{\alpha}{}_{\beta}, \overset{(\sim)}{q}{}^{\gamma'}_{\gamma}\right] = -\delta^{\alpha}_{\gamma}\overset{(\sim)}{q}{}^{\gamma'}_{\beta} + \frac{1}{2}\delta^{\alpha}_{\beta}\overset{(\sim)}{q}{}^{\gamma'}_{\gamma} \qquad \left[\overset{(\sim)}{m}{}^{\alpha'}{}_{\beta'}, \overset{(\sim)}{q}{}^{\gamma'}_{\gamma}\right] = \delta^{\gamma'}_{\beta'}\overset{(\sim)}{q}{}^{\alpha'}_{\gamma} - \frac{1}{2}\delta^{\alpha'}_{\beta'}\overset{(\sim)}{q}{}^{\gamma'}_{\gamma} \tag{B.2.2}$$

$$\left[\overset{(\sim)}{m}{}^{\alpha'}{}_{\beta'}, \overset{(\sim)}{q}{}^{\gamma}_{\gamma'}\right] = -\delta^{\alpha'}_{\gamma'}\overset{(\sim)}{q}{}^{\gamma}_{\beta'} + \frac{1}{2}\delta^{\alpha'}_{\beta'}\overset{(\sim)}{q}{}^{\gamma}_{\gamma'} \qquad \left[\overset{(\sim)}{m}{}^{\alpha}{}_{\beta}, \overset{(\sim)}{q}{}^{\gamma}_{\gamma'}\right] = \delta^{\gamma}_{\beta}\overset{(\sim)}{q}{}^{\alpha}_{\gamma'} - \frac{1}{2}\delta^{\alpha}_{\beta}\overset{(\sim)}{q}{}^{\gamma}_{\gamma'} \tag{B.2.3}$$

as well as

$$\left\{\overset{(\sim)}{q}{}^{\alpha'}_{\alpha}, \overset{(\sim)}{q}{}^{\beta}_{\beta'}\right\} = \pm i(\delta^{\alpha'}_{\beta'}\overset{(\sim)}{m}{}^{\beta}{}_{\alpha} + \delta^{\beta}_{\alpha}\overset{(\sim)}{m}{}^{\alpha'}{}_{\beta'}) \tag{B.2.4}$$

The sign is conventional and will be chosen differently for m, q and $\tilde{m}, \tilde{q}$. [18]

B.3 The light cone basis

The generators in the light cone basis are the translations $P^{\pm}$, conformal boosts $K^{\pm}$, Lorentz rotation J^{+-}, dilatation D, R-symmetry generators $J^{i}{}_{j}$ and $\tilde{J}^{i}{}_{j}$, Poincare algebra supercharges $Q^{\pm i}$ and conformal algebra supercharges $S^{\pm i}$. Their Hermitian conjugates are given by

$$P^{\pm\dagger} = P^{\pm} \qquad J^{+-\dagger} = -J^{+-} \qquad K^{\pm\dagger} = K^{\pm} \qquad D^{\dagger} = -D \tag{B.3.1a}$$

$$Q^{\pm i\dagger} = Q^{\pm}_{i} \qquad S^{\pm i\dagger} = S^{\pm}_{i} \qquad J^{i}{}_{j}{}^{\dagger} = J^{j}{}_{i} \qquad \tilde{J}^{i}{}_{j}{}^{\dagger} = \tilde{J}^{j}{}_{i} \tag{B.3.1b}$$

For the bosonic generators we have

$$\left[P^{\pm}, K^{\mp}\right] = D \mp J^{+-} \tag{B.3.2a}$$

$$\left[D, P^{\pm}\right] = -P^{\pm} \qquad \left[J^{+-}, P^{\pm}\right] = \pm P^{\pm} \tag{B.3.2b}$$

$$\left[D, K^{\pm}\right] = K^{\pm} \qquad \left[J^{+-}, K^{\pm}\right] = \pm K^{\pm} \tag{B.3.2c}$$

$$\left[J^{i}{}_{j}, J^{k}{}_{l}\right] = \delta^{k}_{j}J^{i}{}_{l} - \delta^{i}_{l}J^{k}{}_{j} \qquad \left[\tilde{J}^{i}{}_{j}, \tilde{J}^{k}{}_{l}\right] = \delta^{k}_{j}\tilde{J}^{i}{}_{l} - \delta^{i}_{l}\tilde{J}^{k}{}_{j} \tag{B.3.2d}$$

Under conformal symmetry transformations, the fermionic generators transform as

$$\left[D, Q_i^{\pm}\right] = -\frac{1}{2}Q_i^{\pm} \qquad \left[D, S_i^{\pm}\right] = \frac{1}{2}S_i^{\pm} \tag{B.3.2e}$$

$$\left[J^{+-}, Q_i^{\pm}\right] = \pm\frac{1}{2}Q_i^{\pm} \qquad \left[J^{+-}, S_i^{\pm}\right] = \pm\frac{1}{2}S_i^{\pm} \tag{B.3.2f}$$

$$\left[S_i^{\mp}, P^{\pm}\right] = Q_i^{\pm} \qquad \left[Q_i^{\mp}, K^{\pm}\right] = S_i^{\pm} \tag{B.3.2g}$$

The R-symmetry generators act as

$$\left[J^i{}_j, Q^{-k}\right] = \delta_j^k Q^{-i} - \frac{1}{2}\delta_j^i Q^{-k} \qquad \left[J^i{}_j, Q_k^{-}\right] = -\delta_k^i Q_j^{-} + \frac{1}{2}\delta_j^i Q_k^{-} \tag{B.3.2h}$$

$$\left[J^i{}_j, S^{+k}\right] = \delta_j^k S^{+i} - \frac{1}{2}\delta_j^i S^{+k} \qquad \left[J^i{}_j, S_k^{+}\right] = -\delta_k^i S_j^{+} + \frac{1}{2}\delta_j^i S_k^{+} \tag{B.3.2i}$$

$$\left[\tilde{J}^i{}_j, Q^{+k}\right] = \delta_j^k Q^{+i} - \frac{1}{2}\delta_j^i Q^{+k} \qquad \left[\tilde{J}^i{}_j, Q_k^{+}\right] = -\delta_k^i Q_j^{+} + \frac{1}{2}\delta_j^i Q_k^{+} \tag{B.3.2j}$$

$$\left[\tilde{J}^i{}_j, S^{-k}\right] = \delta_j^k S^{-i} - \frac{1}{2}\delta_j^i S^{-k} \qquad \left[\tilde{J}^i{}_j, S_k^{-}\right] = -\delta_k^i S_j^{-} + \frac{1}{2}\delta_j^i S_k^{-} \tag{B.3.2k}$$

Finally, the supersymmetry generators satisfy

$$\left\{Q^{\pm i}, Q_j^{\pm}\right\} = \pm P^{\pm}\delta_j^i \qquad \left\{Q^{+i}, S_j^{-}\right\} = \frac{1}{2}(J^{+-} - D)\delta_j^i - \tilde{J}^i{}_j \tag{B.3.2l}$$

$$\left\{S^{\pm i}, S_j^{\pm}\right\} = \pm K^{\pm}\delta_j^i \qquad \left\{Q^{-i}, S_j^{+}\right\} = \frac{1}{2}(J^{+-} + D)\delta_j^i + J^i{}_j \tag{B.3.2m}$$

All other (anti)commutators vanish. In this representation the $\mathbb{Z}_4$ grading can also be seen quite easily. Using the notation

$$\mathcal{G}_0 =: \mathcal{H}, \qquad \mathcal{G}_1 =: \mathcal{Q}, \qquad \mathcal{G}_2 =: \mathcal{P}, \qquad \mathcal{G}_3 =: \mathcal{S} \tag{B.3.3}$$

we find that

$$D, J^{+-}, J^i{}_j, \tilde{J}^i{}_j \in \mathcal{H}, \qquad P^{\pm}, K^{\pm} \in \mathcal{P}, \qquad Q^{\pm i} \in \mathcal{Q}, \qquad S^{\pm i} \in \mathcal{S}. \tag{B.3.4}$$

C. CONSTRUCTION OF INVARIANT CHARGES

C.1 Derivation of Maurer-Cartan equations

In the following derivation of nonlocal charges the light cone basis defined in B.3 will be used. Let G denote the Lie supergroup of $\mathcal{G}$ and $g(x)$ a field taking values in G. From the Lagrangian $L \propto \mathrm{Tr}(\partial_i g^{-1}\partial^i g)$ we obtain a global left and right multiplication symmetry. The conserved currents corresponding to left and right multiplication [19]

$$j = -(\partial g)g^{-1} \qquad\qquad J = -g^{-1}\partial g \tag{C.1.1}$$

take values in $\mathcal{G}$. Writing them as one-forms, we see that

$$\mathrm{d}j + j \wedge j = \mathrm{d}J - J \wedge J = 0 \tag{C.1.2}$$

We can make use of the $\mathbb{Z}_4$ grading to write

$$J = H + P + Q + S \tag{C.1.3}$$

where

$$H = L_D D + L^{-+}J^{+-} + L^i{}_j J^j{}_i + \tilde{L}^i{}_j \tilde{J}^j{}_i \tag{C.1.4a}$$

$$P = L_P^- P^+ + L_P^+ P^- + L_K^- K^+ + L_K^+ K^- \tag{C.1.4b}$$

$$Q = L_Q^{-i} Q_i^+ + L_{Qi}^- Q^{+i} + L_Q^{+i} Q_i^- + L_{Qi}^+ Q^{-i} \tag{C.1.4c}$$

$$S = L_S^{-i} S_i^+ + L_{Si}^- S^{+i} + L_S^{+i} S_i^- + L_{Si}^+ S^{-i} \tag{C.1.4d}$$

and the L's are Cartan 1-forms. As we have already seen, the curl $\mathrm{d}J = J \wedge J$ also decomposes according to the $\mathbb{Z}_4$ grading. This simplifies the computation of the Maurer-Cartan equations. Using

$$H \wedge H = -L^i{}_k \wedge L^k{}_j J^j{}_i + (\tilde{\ldots}) \tag{C.1.5a}$$

$$P \wedge P = L_P^- \wedge L_K^+(D - J^{+-}) + L_P^+ \wedge L_K^-(D + J^{+-}) \tag{C.1.5b}$$

$$\begin{aligned} Q \wedge S + S \wedge Q = {} & L_Q^{-i} \wedge L_{Sj}^+ \left(-\frac{1}{2}(J^{+-} - D)\delta_i^j - \tilde{J}^j{}_i\right) + L_{Qi}^- \wedge L_S^{+j}\left(\frac{1}{2}(J^{+-} - D)\delta_j^i - \tilde{J}^i{}_j\right) \\ & + L_Q^{+i} \wedge L_{Sj}^- \left(-\frac{1}{2}(J^{+-} + D)\delta_i^j + J^j{}_i\right) + L_{Qi}^+ \wedge L_S^{-j}\left(\frac{1}{2}(J^{+-} + D)\delta_j^i + J^i{}_j\right) \end{aligned} \tag{C.1.5c}$$

$$\begin{aligned} H \wedge P + P \wedge H = {} & -L_D \wedge L_P^- P^+ - L_D \wedge L_P^+ P^- + L_D \wedge L_K^- K^+ + L_D \wedge L_K^+ K^- \\ & + L^{-+} \wedge L_P^- P^+ - L^{-+} \wedge L_P^+ P^- + L^{-+} \wedge L_K^- K^+ - L^{-+} \wedge L_K^+ K^- \end{aligned} \tag{C.1.5d}$$

$$Q \wedge Q = L_Q^{-i} \wedge L_{Qi}^{-} P^{+} - L_Q^{+i} \wedge L_{Qi}^{+} P^{-} \tag{C.1.5e}$$

$$S \wedge S = L_S^{-i} \wedge L_{Si}^{-} K^{+} - L_S^{+i} \wedge L_{Si}^{+} K^{-} \tag{C.1.5f}$$

$$\begin{aligned} H \wedge Q + Q \wedge H = &\left(-\frac{1}{2} L_D \wedge L_Q^{-i} + \frac{1}{2} L^{-+} \wedge L_Q^{-i} - \tilde{L}^i{}_j \wedge L_Q^{-j}\right) Q_i^{+} \\ &+ \left(-\frac{1}{2} L_D \wedge L_{Qi}^{-} + \frac{1}{2} L^{-+} \wedge L_{Qi}^{-} + \tilde{L}^j{}_i \wedge L_{Qj}^{-}\right) Q^{+i} \\ &+ \left(-\frac{1}{2} L_D \wedge L_Q^{+i} - \frac{1}{2} L^{-+} \wedge L_Q^{+i} - L^i{}_j \wedge L_Q^{+j}\right) Q_i^{-} \\ &+ \left(-\frac{1}{2} L_D \wedge L_{Qi}^{+} - \frac{1}{2} L^{-+} \wedge L_{Qi}^{+} + L^j{}_i \wedge L_{Qj}^{+}\right) Q^{-i} \end{aligned} \tag{C.1.5g}$$

$$P \wedge S + S \wedge P = -L_P^{+} \wedge L_S^{-i} Q_i^{-} + L_P^{+} \wedge L_{Si}^{-} Q^{-i} - L_P^{-} \wedge L_S^{+i} Q_i^{+} + L_P^{-} \wedge L_{Si}^{+} Q^{+i} \tag{C.1.5h}$$

$$\begin{aligned} H \wedge S + S \wedge H = &\left(\frac{1}{2} L_D \wedge L_S^{-i} + \frac{1}{2} L^{-+} \wedge L_S^{-i} - L^i{}_j \wedge L_S^{-j}\right) S_i^{+} \\ &+ \left(\frac{1}{2} L_D \wedge L_{Si}^{-} + \frac{1}{2} L^{-+} \wedge L_{Si}^{-} + L^j{}_i \wedge L_{Sj}^{-}\right) S^{+i} \\ &+ \left(\frac{1}{2} L_D \wedge L_S^{+i} - \frac{1}{2} L^{-+} \wedge L_S^{+i} - \tilde{L}^i{}_j \wedge L_S^{+j}\right) S_i^{-} \\ &+ \left(\frac{1}{2} L_D \wedge L_{Si}^{+} - \frac{1}{2} L^{-+} \wedge L_{Si}^{+} + \tilde{L}^j{}_i \wedge L_{Sj}^{+}\right) S^{-i} \end{aligned} \tag{C.1.5i}$$

$$P \wedge Q + Q \wedge P = -L_K^{+} \wedge L_Q^{-i} S_i^{-} + L_K^{+} \wedge L_{Qi}^{-} S^{-i} - L_K^{-} \wedge L_Q^{+i} S_i^{+} + L_K^{-} \wedge L_{Qi}^{+} S^{+i} \tag{C.1.5j}$$

we can easily deduce the Maurer-Cartan equations

$$\mathrm{d}L_D = L_P^- \wedge L_K^+ + L_P^+ \wedge L_K^- + \frac{1}{2}(-L_Q^{+i} \wedge L_{Si}^- + L_{Qi}^+ \wedge L_S^{-i} + L_Q^{-i} \wedge L_{Si}^+ - L_{Qi}^- \wedge L_S^{+i}) \tag{C.1.6a}$$

$$\mathrm{d}L^{-+} = -L_P^- \wedge L_K^+ + L_P^+ \wedge L_K^- + \frac{1}{2}(-L_Q^{+i} \wedge L_{Si}^- + L_{Qi}^+ \wedge L_S^{-i} - L_Q^{-i} \wedge L_{Si}^+ + L_{Qi}^- \wedge L_S^{+i}) \tag{C.1.6b}$$

$$\mathrm{d}L^i{}_j = -L^i{}_k \wedge L^k{}_j + L_Q^{+i} \wedge L_{Sj}^- + L_{Qj}^+ \wedge L_S^{-i} \tag{C.1.6c}$$

$$\mathrm{d}\tilde{L}^i{}_j = -\tilde{L}^i{}_k \wedge \tilde{L}^k{}_j - L_Q^{-i} \wedge L_{Sj}^+ - L_{Qj}^- \wedge L_S^{+i} \tag{C.1.6d}$$

$$\mathrm{d}L_P^\pm = -L_D \wedge L_P^\pm \mp L^{-+} \wedge L_P^\pm \mp L_Q^{\pm i} \wedge L_{Qi}^\pm \tag{C.1.6e}$$

$$\mathrm{d}L_K^\pm = L_D \wedge L_K^\pm \mp L^{-+} \wedge L_K^\pm \mp L_S^{\pm i} \wedge L_{Si}^\pm \tag{C.1.6f}$$

$$\mathrm{d}L_Q^{-i} = \frac{1}{2}(-L_D + L^{-+}) \wedge L_Q^{-i} - \tilde{L}^i{}_j \wedge L_Q^{-j} - L_P^- \wedge L_S^{+i} \tag{C.1.6g}$$

$$\mathrm{d}L_{Qi}^- = \frac{1}{2}(-L_D + L^{-+}) \wedge L_{Qi}^- + \tilde{L}^j{}_i \wedge L_{Qj}^- + L_P^- \wedge L_{Si}^+ \tag{C.1.6h}$$

$$\mathrm{d}L_Q^{+i} = \frac{1}{2}(-L_D - L^{-+}) \wedge L_Q^{+i} - L^i{}_j \wedge L_Q^{+j} - L_P^+ \wedge L_S^{-i} \tag{C.1.6i}$$

$$\mathrm{d}L_{Qi}^+ = \frac{1}{2}(-L_D - L^{-+}) \wedge L_{Qi}^+ + L^j{}_i \wedge L_{Qj}^+ + L_P^+ \wedge L_{Si}^- \tag{C.1.6j}$$

$$\mathrm{d}L_S^{-i} = \frac{1}{2}(L_D + L^{-+}) \wedge L_S^{-i} - L^i{}_j \wedge L_S^{-j} - L_K^- \wedge L_Q^{+i} \tag{C.1.6k}$$

$$\mathrm{d}L_{Si}^- = \frac{1}{2}(L_D + L^{-+}) \wedge L_{Si}^- + L^j{}_i \wedge L_{Sj}^- + L_K^- \wedge L_{Qi}^+ \tag{C.1.6l}$$

$$\mathrm{d}L_S^{+i} = \frac{1}{2}(L_D - L^{-+}) \wedge L_S^{+i} - \tilde{L}^i{}_j \wedge L_S^{+j} - L_K^+ \wedge L_Q^{-i} \tag{C.1.6m}$$

$$\mathrm{d}L_{Si}^+ = \frac{1}{2}(L_D - L^{-+}) \wedge L_{Si}^+ + \tilde{L}^j{}_i \wedge L_{Sj}^+ + L_K^+ \wedge L_{Qi}^- \tag{C.1.6n}$$

C.2 Variation of Cartan 1-forms

In order to obtain the conserved charges and the equations of motion, we need to compute the variation of the Cartan 1-forms given above under right multiplication. Let $g' = g(1+\omega)$ and $J' = -g'^{-1}\,\mathrm{d}g'$. Then we have

$$\begin{aligned}
\delta J &= -g'^{-1}\,\mathrm{d}g' + g^{-1}\,\mathrm{d}g \\
&= -(1+\omega)^{-1} g^{-1}\,\mathrm{d}(g(1+\omega)) + g^{-1}\,\mathrm{d}g \\
&= -(1-\omega) g^{-1}(\,\mathrm{d}g\,(1+\omega) + g\,\mathrm{d}\omega\,) + g^{-1}\,\mathrm{d}g \\
&= \omega g^{-1}\,\mathrm{d}g - g^{-1}\,\mathrm{d}g\,\omega - \mathrm{d}\omega \\
&= -\,\mathrm{d}\omega - \left[g^{-1}\,\mathrm{d}g\,,\omega\right] \\
&= -\,\mathrm{d}\omega + [J,\omega]
\end{aligned} \tag{C.2.1}$$

For an explicit computation, we use the expansion

$$\begin{aligned}
\omega = {}& \omega_D D + \omega^{-+} J^{+-} + \omega^i{}_j J^j{}_i + \tilde{\omega}^i{}_j \tilde{J}^j{}_i + \omega_P^- P^+ + \omega_P^+ P^- + \omega_K^- K^+ + \omega_K^+ K^- \\
& + \omega_Q^{-i} Q_i^+ + \omega_{Qi}^- Q^{+i} + \omega_Q^{+i} Q_i^- + \omega_{Qi}^+ Q^{-i} + \omega_S^{-i} S_i^+ + \omega_{Si}^- S^{+i} + \omega_S^{+i} S_i^- + \omega_{Si}^+ S^{-i}
\end{aligned} \tag{C.2.2}$$

From this we obtain

$$
\begin{aligned}
\delta J =& -\mathrm{d}\omega \ -(L_D\omega_P^- - L_P^-\omega_D)P^+ - (L_D\omega_P^+ - L_P^+\omega_D)P^- + (L_D\omega_K^- - L_K^-\omega_D)K^+ + (L_D\omega_K^+ - L_K^+\omega_D)K^- \\
&- \frac{1}{2}\Big((L_D\omega_Q^{-i} - L_Q^{-i}\omega_D)Q_i^+ + (L_D\omega_Q^{+i} - L_Q^{+i}\omega_D)Q_i^- + (L_D\omega_{Qi}^- - L_{Qi}^-\omega_D)Q^{+i} + (L_D\omega_{Qi}^+ - L_{Qi}^+\omega_D)Q^{-i}\Big) \\
&+ \frac{1}{2}\left((L_D\omega_S^{-i} - L_S^{-i}\omega_D)S_i^+ + (L_D\omega_S^{+i} - L_S^{+i}\omega_D)S_i^- + (L_D\omega_{Si}^- - L_{Si}^-\omega_D)S^{+i} + (L_D\omega_{Si}^+ - L_{Si}^+\omega_D)S^{-i}\right) \\
&+ (L^{-+}\omega_P^- - L_P^-\omega^{-+})P^+ - (L^{-+}\omega_P^+ - L_P^+\omega^{-+})P^- + (L^{-+}\omega_K^- - L_K^-\omega^{-+})K^+ - (L^{-+}\omega_K^+ - L_K^+\omega^{-+})K^- \\
&+ \frac{1}{2}\Big((L^{-+}\omega_Q^{-i} - L_Q^{-i}\omega^{-+})Q_i^+ - (L^{-+}\omega_Q^{+i} - L_Q^{+i}\omega^{-+})Q_i^- + (L^{-+}\omega_{Qi}^- - L_{Qi}^-\omega^{-+})Q^{+i} - (L^{-+}\omega_{Qi}^+ - L_{Qi}^+\omega^{-+})Q^{-i}\Big) \\
&+ \frac{1}{2}\left((L^{-+}\omega_S^{-i} - L_S^{-i}\omega^{-+})S_i^+ - (L^{-+}\omega_S^{+i} - L_S^{+i}\omega^{-+})S_i^- + (L^{-+}\omega_{Si}^- - L_{Si}^-\omega^{-+})S^{+i} - (L^{-+}\omega_{Si}^+ - L_{Si}^+\omega^{-+})S^{-i}\right) \\
&+ L^j{}_i\omega^l{}_k(\delta_j^k J^i{}_l - \delta_l^i J^k{}_j) + \tilde{L}^j{}_i\tilde{\omega}^l{}_k(\delta_j^k \tilde{J}^i{}_l - \delta_l^i \tilde{J}^k{}_j) \\
&+ (L^j{}_i\omega_{Qk}^+ - L_{Qk}^+\omega^j{}_i)\left(\delta_j^k Q^{-i} - \frac{1}{2}\delta_j^i Q^{-k}\right) + (L^j{}_i\omega_Q^{+k} - L_Q^{+k}\omega^j{}_i)\left(-\delta_k^i Q_j^- + \frac{1}{2}\delta_j^i Q_k^-\right) \\
&+ (L^j{}_i\omega_{Sk}^- - L_{Sk}^-\omega^j{}_i)\left(\delta_j^k S^{+i} - \frac{1}{2}\delta_j^i S^{+k}\right) + (L^j{}_i\omega_S^{-k} - L_S^{-k}\omega^j{}_i)\left(-\delta_k^i S_j^+ + \frac{1}{2}\delta_j^i S_k^+\right) \\
&+ (\tilde{L}^j{}_i\omega_{Qk}^- - L_{Qk}^-\tilde{\omega}^j{}_i)\left(\delta_j^k Q^{+i} - \frac{1}{2}\delta_j^i Q^{+k}\right) + (\tilde{L}^j{}_i\omega_Q^{-k} - L_Q^{-k}\tilde{\omega}^j{}_i)\left(-\delta_k^i Q_j^+ + \frac{1}{2}\delta_j^i Q_k^+\right) \\
&+ (\tilde{L}^j{}_i\omega_{Sk}^+ - L_{Sk}^+\tilde{\omega}^j{}_i)\left(\delta_j^k S^{-i} - \frac{1}{2}\delta_j^i S^{-k}\right) + (\tilde{L}^j{}_i\omega_S^{+k} - L_S^{+k}\tilde{\omega}^j{}_i)\left(-\delta_k^i S_j^- + \frac{1}{2}\delta_j^i S_k^-\right) \\
&+ (L_P^-\omega_K^+ - L_K^+\omega_P^-)(D - J^{+-}) + (L_P^+\omega_K^- - L_K^-\omega_P^+)(D + J^{+-}) \\
&- (L_P^-\omega_S^{+i} - L_S^{+i}\omega_P^-)Q_i^+ - (L_P^+\omega_S^{-i} - L_S^{-i}\omega_P^+)Q_i^- + (L_P^-\omega_{Si}^+ - L_{Si}^+\omega_P^-)Q^{+i} + (L_P^+\omega_{Si}^- - L_{Si}^-\omega_P^+)Q^{-i} \\
&- (L_K^-\omega_Q^{+i} - L_Q^{+i}\omega_K^-)S_i^+ - (L_K^+\omega_Q^{-i} - L_Q^{-i}\omega_K^+)S_i^- + (L_K^-\omega_{Qi}^+ - L_{Qi}^+\omega_K^-)S^{+i} + (L_K^+\omega_{Qi}^- - L_{Qi}^-\omega_K^+)S^{-i} \\
&- (L_{Qi}^-\omega_Q^{-j} + L_Q^{-j}\omega_{Qi}^-)P^+\delta_j^i + (L_{Qi}^+\omega_Q^{+j} + L_Q^{+j}\omega_{Qi}^+)P^-\delta_j^i - (L_{Si}^-\omega_S^{-j} + L_S^{-j}\omega_{Si}^-)K^+\delta_j^i + (L_{Si}^+\omega_S^{+j} + L_S^{+j}\omega_{Si}^+)K^-\delta_j^i \\
&- (L_{Qi}^-\omega_S^{+j} + L_S^{+j}\omega_{Qi}^-)\left(\frac{1}{2}(J^{+-} - D)\delta_j^i - \tilde{J}^i{}_j\right) - (L_{Qi}^+\omega_S^{-j} + L_S^{-j}\omega_{Qi}^+)\left(\frac{1}{2}(J^{+-} + D)\delta_j^i + J^i{}_j\right) \\
&- (L_Q^{-i}\omega_{Sj}^+ + L_{Sj}^+\omega_Q^{-i})\left(-\frac{1}{2}(J^{+-} - D)\delta_i^j - \tilde{J}^j{}_i\right) - (L_Q^{+i}\omega_{Sj}^- + L_{Sj}^-\omega_Q^{+i})\left(-\frac{1}{2}(J^{+-} + D)\delta_i^j + J^j{}_i\right)
\end{aligned}
\tag{C.2.3}
$$

This can be decomposed into

$$
\begin{aligned}
\delta L_D =& -\mathrm{d}\omega_D \ + L_P^-\omega_K^+ - L_K^+\omega_P^- + L_P^+\omega_K^- - L_K^-\omega_P^+ \\
&+ \frac{1}{2}\left(L_{Qi}^-\omega_S^{+i} + L_S^{+i}\omega_{Qi}^- - L_{Qi}^+\omega_S^{-i} - L_S^{-i}\omega_{Qi}^+ - L_Q^{-i}\omega_{Si}^+ - L_{Si}^+\omega_Q^{-i} + L_Q^{+i}\omega_{Si}^- + L_{Si}^-\omega_Q^{+i}\right)
\end{aligned}
\tag{C.2.4a}
$$

$$
\begin{aligned}
\delta L^{-+} =& -\mathrm{d}\omega^{-+} \ - L_P^-\omega_K^+ + L_K^+\omega_P^- + L_P^+\omega_K^- - L_K^-\omega_P^+ \\
&+ \frac{1}{2}\left(-L_{Qi}^-\omega_S^{+i} - L_S^{+i}\omega_{Qi}^- - L_{Qi}^+\omega_S^{-i} - L_S^{-i}\omega_{Qi}^+ + L_Q^{-i}\omega_{Si}^+ + L_{Si}^+\omega_Q^{-i} + L_Q^{+i}\omega_{Si}^- + L_{Si}^-\omega_Q^{+i}\right)
\end{aligned}
\tag{C.2.4b}
$$

$$
\delta L^i{}_j = -\mathrm{d}\omega^i{}_j \ + L^k{}_j\omega^i{}_k - L^i{}_k\omega^k{}_j - L_{Qj}^+\omega_S^{-i} - L_S^{-i}\omega_{Qj}^+ - L_Q^{+i}\omega_{Sj}^- - L_{Sj}^-\omega_Q^{+i} \tag{C.2.4c}
$$

$$
\delta \tilde{L}^i{}_j = -\mathrm{d}\tilde{\omega}^i{}_j \ + \tilde{L}^k{}_j\tilde{\omega}^i{}_k - \tilde{L}^i{}_k\tilde{\omega}^k{}_j + L_{Qj}^-\omega_S^{+i} + L_S^{+i}\omega_{Qj}^- + L_Q^{-i}\omega_{Sj}^+ + L_{Sj}^+\omega_Q^{-i} \tag{C.2.4d}
$$

$$\delta L_P^{\pm} = -\,\mathrm{d}\omega_P^{\pm} \; - L_D\omega_P^{\pm} + L_P^{\pm}\omega_D \mp L^{-+}\omega_P^{\pm} \pm L_P^{\pm}\omega^{-+} \pm L_{Qi}^{\pm}\omega_Q^{\pm i} \pm L_Q^{\pm i}\omega_{Qi}^{\pm} \tag{C.2.4e}$$

$$\delta L_K^{\pm} = -\,\mathrm{d}\omega_K^{\pm} \; + L_D\omega_K^{\pm} - L_K^{\pm}\omega_D \mp L^{-+}\omega_K^{\pm} \pm L_K^{\pm}\omega^{-+} \pm L_{Si}^{\pm}\omega_S^{\pm i} \pm L_S^{\pm i}\omega_{Si}^{\pm} \tag{C.2.4f}$$

$$\delta L_Q^{-i} = -\,\mathrm{d}\omega_Q^{-i} \; + \frac{1}{2}((L^{-+} - L_D)\omega_Q^{-i} - L_Q^{-i}(\omega^{-+} - \omega_D)) - \tilde{L}^i{}_j\omega_Q^{-j} + L_Q^{-j}\tilde{\omega}^i{}_j - L_P^{-}\omega_S^{+i} + L_S^{+i}\omega_P^{-} \tag{C.2.4g}$$

$$\delta L_{Qi}^{-} = -\,\mathrm{d}\omega_{Qi}^{-} \; + \frac{1}{2}((L^{-+} - L_D)\omega_{Qi}^{-} - L_{Qi}^{-}(\omega^{-+} - \omega_D)) + \tilde{L}^j{}_i\omega_{Qj}^{-} - L_{Qj}^{-}\tilde{\omega}^j{}_i + L_P^{-}\omega_{Si}^{+} - L_{Si}^{+}\omega_P^{-} \tag{C.2.4h}$$

$$\delta L_Q^{+i} = -\,\mathrm{d}\omega_Q^{+i} + \frac{1}{2}((-L^{-+} - L_D)\omega_Q^{+i} - L_Q^{+i}(-\omega^{-+} - \omega_D)) - L^i{}_j\omega_Q^{+j} + L_Q^{+j}\omega^i{}_j - L_P^{+}\omega_S^{-i} + L_S^{-i}\omega_P^{+} \tag{C.2.4i}$$

$$\delta L_{Qi}^{+} = -\,\mathrm{d}\omega_{Qi}^{+} + \frac{1}{2}((-L^{-+} - L_D)\omega_{Qi}^{+} - L_{Qi}^{+}(-\omega^{-+} - \omega_D)) + L^j{}_i\omega_{Qj}^{+} - L_{Qj}^{+}\omega^j{}_i + L_P^{+}\omega_{Si}^{-} - L_{Si}^{-}\omega_P^{+} \tag{C.2.4j}$$

$$\delta L_S^{-i} = -\,\mathrm{d}\omega_S^{-i} \; + \frac{1}{2}((L^{-+} + L_D)\omega_S^{-i} - L_S^{-i}(\omega^{-+} + \omega_D)) - L^i{}_j\omega_S^{-j} + L_S^{-j}\omega^i{}_j - L_K^{-}\omega_Q^{+i} + L_Q^{+i}\omega_K^{-} \tag{C.2.4k}$$

$$\delta L_{Si}^{-} = -\mathrm{d}\omega_{Si}^{-} + \frac{1}{2}((L^{-+} + L_D)\omega_{Si}^{-} - L_{Si}^{-}(\omega^{-+} + \omega_D)) + L^j{}_i\omega_{Sj}^{-} - L_{Sj}^{-}\omega^j{}_i + L_K^{-}\omega_{Qi}^{+} - L_{Qi}^{+}\omega_K^{-} \tag{C.2.4l}$$

$$\delta L_S^{+i} = -\,\mathrm{d}\omega_S^{+i} + \frac{1}{2}((-L^{-+} + L_D)\omega_S^{+i} - L_S^{+i}(-\omega^{-+} + \omega_D)) - \tilde{L}^i{}_j\omega_S^{+j} + L_S^{+j}\tilde{\omega}^i{}_j - L_K^{+}\omega_Q^{-i} + L_Q^{-i}\omega_K^{+} \tag{C.2.4m}$$

$$\delta L_{Si}^{+} = -\,\mathrm{d}\omega_{Si}^{+} \; + \frac{1}{2}((-L^{-+} + L_D)\omega_{Si}^{+} - L_{Si}^{+}(-\omega^{-+} + \omega_D)) + \tilde{L}^j{}_i\omega_{Sj}^{+} - L_{Sj}^{+}\tilde{\omega}^j{}_i + L_K^{+}\omega_{Qi}^{-} - L_{Qi}^{-}\omega_K^{+} \tag{C.2.4n}$$

C.3 Supersymmetric string action

The Lagrangian consists of a kinetic term and a Wess-Zumino term, which can be written as

$$\mathcal{L}_{kin} = -\frac{1}{2}\sqrt{g}g^{\mu\nu}(\hat{L}_\mu^a\hat{L}_\nu^a + L_{D\mu}L_{D\nu} + L_\mu^{A'}L_\nu^{A'}) \tag{C.3.1a}$$

$$\mathcal{L}_{WZ} = -\frac{i}{\sqrt{2}}\epsilon^{\mu\nu}L_{Q\mu}^{+i}c\epsilon_{ij}L_{Q\nu}^{-j} + \text{h.c.} \tag{C.3.1b}$$

where

$$\hat{L}^a = L_P^a - \frac{1}{2}L_K^a, \qquad L^{A'} = \frac{i}{2}(\sigma^{A'})^i{}_j(\tilde{L}^j{}_i - L^j{}_i) \tag{C.3.2}$$

and $c\epsilon_{ij}$ is a charge conjugation matrix with $|c| = 1$. Let $\sigma^i{}_j{}^k{}_l = (\sigma^{A'})^i{}_j(\sigma^{A'})^k{}_l$. The variation of this Lagrangian is given by

$$
\begin{aligned}
\delta\mathcal{L}_{kin} &= -\sqrt{g}g^{\mu\nu}(\hat{L}^a_\mu\delta\hat{L}^a_\nu + L_{D\mu}\delta L_{D\nu} + L^{A'}_\mu\delta L^{A'}_\nu) \\
&= -\sqrt{g}g^{\mu\nu}\left(L^a_{P\mu} - \frac{1}{2}L^a_{K\mu}\right)\left(\delta L^a_{P\nu} - \frac{1}{2}\delta L^a_{K\nu}\right) - \sqrt{g}g^{\mu\nu}L_{D\mu}\delta L_{D\nu} \\
&\quad + \frac{1}{4}\sqrt{g}g^{\mu\nu}\sigma^l{}_k{}^j{}_i(\tilde{L}^k{}_{l\mu} - L^k{}_{l\mu})(\delta\tilde{L}^i{}_{j\nu} - \delta L^i{}_{j\nu}) \\
&= -\sqrt{g}g^{\mu\nu}\left(L^+_{P\mu} - \frac{1}{2}L^+_{K\mu}\right)\left(-\,\mathrm{d}\omega^-_{P\nu} - L_{D\nu}\omega^-_P + L^-_{P\nu}\omega_D + L^{-+}_\nu\omega^-_P - L^-_{P\nu}\omega^{-+} - L^-_{Qi\nu}\omega^{-i}_Q - L^{-i}_{Q\nu}\omega^-_{Qi}\right) \\
&\quad + \frac{1}{2}\sqrt{g}g^{\mu\nu}\left(L^+_{P\mu} - \frac{1}{2}L^+_{K\mu}\right)\left(-\,\mathrm{d}\omega^-_{K\nu} + L_{D\nu}\omega^-_K - L^-_{K\nu}\omega_D + L^{-+}_\nu\omega^-_K - L^-_{K\nu}\omega^{-+} - L^-_{Si\nu}\omega^{-i}_S - L^{-i}_{S\nu}\omega^-_{Si}\right) \\
&\quad - \sqrt{g}g^{\mu\nu}\left(L^-_{P\mu} - \frac{1}{2}L^-_{K\mu}\right)\left(-\,\mathrm{d}\omega^+_{P\nu} - L_{D\nu}\omega^+_P + L^+_{P\nu}\omega_D - L^{-+}_\nu\omega^+_P + L^+_{P\nu}\omega^{-+} + L^+_{Qi\nu}\omega^{+i}_Q + L^{+i}_{Q\nu}\omega^+_{Qi}\right) \\
&\quad + \frac{1}{2}\sqrt{g}g^{\mu\nu}\left(L^-_{P\mu} - \frac{1}{2}L^-_{K\mu}\right)\left(-\,\mathrm{d}\omega^+_{K\nu} + L_{D\nu}\omega^+_K - L^+_{K\nu}\omega_D - L^{-+}_\nu\omega^+_K + L^+_{K\nu}\omega^{-+} + L^+_{Si\nu}\omega^{+i}_S + L^{+i}_{S\nu}\omega^+_{Si}\right) \\
&\quad - \sqrt{g}g^{\mu\nu}L_{D\mu}\left(-\,\mathrm{d}\omega_{D\nu} + L^-_{P\nu}\omega^+_K - L^+_{K\nu}\omega^-_P + L^+_{P\nu}\omega^-_K - L^-_{K\nu}\omega^+_P\right) \\
&\quad - \frac{1}{2}\sqrt{g}g^{\mu\nu}L_{D\mu}\left(L^-_{Qi\nu}\omega^{+i}_S + L^{+i}_{S\nu}\omega^-_{Qi} - L^+_{Qi\nu}\omega^{-i}_S - L^{-i}_{S\nu}\omega^+_{Qi} - L^{-i}_{Q\nu}\omega^+_{Si} - L^+_{Si\nu}\omega^{-i}_Q + L^{+i}_{Q\nu}\omega^-_{Si} + L^-_{Si\nu}\omega^{+i}_Q\right) \\
&\quad + \frac{1}{4}\sqrt{g}g^{\mu\nu}\sigma^l{}_k{}^j{}_i(\tilde{L}^k{}_{l\mu} - L^k{}_{l\mu})\left(-\,\mathrm{d}\tilde{\omega}^i{}_{j\nu} + \tilde{L}^m{}_{j\nu}\tilde{\omega}^i{}_m - \tilde{L}^i{}_{m\nu}\tilde{\omega}^m{}_j + L^-_{Qj\nu}\omega^{+i}_S + L^{+i}_{S\nu}\omega^-_{Qj} + L^{-i}_{Q\nu}\omega^+_{Sj} + L^+_{Sj\nu}\omega^{-i}_Q\right) \\
&\quad - \frac{1}{4}\sqrt{g}g^{\mu\nu}\sigma^l{}_k{}^j{}_i(\tilde{L}^k{}_{l\mu} - L^k{}_{l\mu})\left(-\,\mathrm{d}\omega^i{}_{j\nu} + L^m{}_{j\nu}\omega^i{}_m - L^i{}_{m\nu}\omega^m{}_j - L^+_{Qj\nu}\omega^{-i}_S - L^{-i}_{S\nu}\omega^+_{Qj} - L^{+i}_{Q\nu}\omega^-_{Sj} - L^-_{Sj\nu}\omega^{+i}_Q\right) \\
&= \partial_\nu\left(\sqrt{g}g^{\mu\nu}\left(\left(L^a_{P\mu} - \frac{1}{2}L^a_{K\mu}\right)\left(\omega^a_P - \frac{1}{2}\omega^a_K\right) + L_{D\mu}\omega_D - \frac{1}{4}\sigma^l{}_k{}^j{}_i(\tilde{L}^k{}_{l\mu} - L^k{}_{l\mu})(\tilde{\omega}^i{}_j - \omega^i{}_j)\right)\right) \\
&\quad - \sqrt{g}g^{\mu\nu}\left(\left(\nabla_\nu L^a_{P\mu} - \frac{1}{2}\nabla_\nu L^a_{K\mu}\right)\left(\omega^a_P - \frac{1}{2}\omega^a_K\right) + \nabla_\nu L_{D\mu}\omega_D - \frac{1}{4}\sigma^l{}_k{}^j{}_i(\nabla_\nu\tilde{L}^k{}_{l\mu} - \nabla_\nu L^k{}_{l\mu})(\tilde{\omega}^i{}_j - \omega^i{}_j)\right) \\
&\quad - \sqrt{g}g^{\mu\nu}\left(L^+_{P\mu} - \frac{1}{2}L^+_{K\mu}\right)\left(-L_{D\nu}\omega^-_P + L^-_{P\nu}\omega_D + L^{-+}_\nu\omega^-_P - L^-_{P\nu}\omega^{-+} - L^-_{Qi\nu}\omega^{-i}_Q - L^{-i}_{Q\nu}\omega^-_{Qi}\right) \\
&\quad + \frac{1}{2}\sqrt{g}g^{\mu\nu}\left(L^+_{P\mu} - \frac{1}{2}L^+_{K\mu}\right)\left(L_{D\nu}\omega^-_K - L^-_{K\nu}\omega_D + L^{-+}_\nu\omega^-_K - L^-_{K\nu}\omega^{-+} - L^-_{Si\nu}\omega^{-i}_S - L^{-i}_{S\nu}\omega^-_{Si}\right) \\
&\quad - \sqrt{g}g^{\mu\nu}\left(L^-_{P\mu} - \frac{1}{2}L^-_{K\mu}\right)\left(-L_{D\nu}\omega^+_P + L^+_{P\nu}\omega_D - L^{-+}_\nu\omega^+_P + L^+_{P\nu}\omega^{-+} + L^+_{Qi\nu}\omega^{+i}_Q + L^{+i}_{Q\nu}\omega^+_{Qi}\right) \\
&\quad + \frac{1}{2}\sqrt{g}g^{\mu\nu}\left(L^-_{P\mu} - \frac{1}{2}L^-_{K\mu}\right)\left(L_{D\nu}\omega^+_K - L^+_{K\nu}\omega_D - L^{-+}_\nu\omega^+_K + L^+_{K\nu}\omega^{-+} + L^+_{Si\nu}\omega^{+i}_S + L^{+i}_{S\nu}\omega^+_{Si}\right) \\
&\quad - \sqrt{g}g^{\mu\nu}L_{D\mu}\left(L^-_{P\nu}\omega^+_K - L^+_{K\nu}\omega^-_P + L^+_{P\nu}\omega^-_K - L^-_{K\nu}\omega^+_P\right) \\
&\quad - \frac{1}{2}\sqrt{g}g^{\mu\nu}L_{D\mu}\left(L^-_{Qi\nu}\omega^{+i}_S + L^{+i}_{S\nu}\omega^-_{Qi} - L^+_{Qi\nu}\omega^{-i}_S - L^{-i}_{S\nu}\omega^+_{Qi} - L^{-i}_{Q\nu}\omega^+_{Si} - L^+_{Si\nu}\omega^{-i}_Q + L^{+i}_{Q\nu}\omega^-_{Si} + L^-_{Si\nu}\omega^{+i}_Q\right) \\
&\quad + \frac{1}{4}\sqrt{g}g^{\mu\nu}\sigma^l{}_k{}^j{}_i(\tilde{L}^k{}_{l\mu} - L^k{}_{l\mu})\left(\tilde{L}^m{}_{j\nu}\tilde{\omega}^i{}_m - \tilde{L}^i{}_{m\nu}\tilde{\omega}^m{}_j + L^-_{Qj\nu}\omega^{+i}_S + L^{+i}_{S\nu}\omega^-_{Qj} + L^{-i}_{Q\nu}\omega^+_{Sj} + L^+_{Sj\nu}\omega^{-i}_Q\right) \\
&\quad - \frac{1}{4}\sqrt{g}g^{\mu\nu}\sigma^l{}_k{}^j{}_i(\tilde{L}^k{}_{l\mu} - L^k{}_{l\mu})\left(L^m{}_{j\nu}\omega^i{}_m - L^i{}_{m\nu}\omega^m{}_j - L^+_{Qj\nu}\omega^{-i}_S - L^{-i}_{S\nu}\omega^+_{Qj} - L^{+i}_{Q\nu}\omega^-_{Sj} - L^-_{Sj\nu}\omega^{+i}_Q\right)
\end{aligned}
\tag{C.3.3a}
$$

$$
\begin{aligned}
\delta\mathcal{L}_{WZ} =& -\frac{i}{\sqrt{2}}\left(\epsilon^{\mu\nu}\delta L^{+i}_{Q\mu}c\epsilon_{ij}L^{-j}_{Q\nu} + \epsilon^{\mu\nu}L^{+i}_{Q\mu}c\epsilon_{ij}\delta L^{-j}_{Q\nu} - \epsilon^{\mu\nu}\delta L^{+}_{Qi\mu}c^*\epsilon^{ij}L^{-}_{Qj\nu} - \epsilon^{\mu\nu}L^{+}_{Qi\mu}c^*\epsilon^{ij}\delta L^{-}_{Qj\nu}\right)\\
=& -\frac{i}{\sqrt{2}}\epsilon^{\mu\nu}c\epsilon_{ij}\left(-\,\mathrm{d}\omega^{+i}_{Q\mu} + \frac{1}{2}((-L^{-+}_{\mu} - L_{D\mu})\omega^{+i}_{Q} - L^{+i}_{Q\mu}(-\omega^{-+} - \omega_D))\right)L^{-j}_{Q\nu}\\
& - \frac{i}{\sqrt{2}}\epsilon^{\mu\nu}c\epsilon_{ij}\left(-L^{i}{}_{k\mu}\omega^{+k}_{Q} + L^{+k}_{Q\mu}\omega^{i}{}_{k} - L^{+}_{P\mu}\omega^{-i}_{S} + L^{-i}_{S\mu}\omega^{+}_{P}\right)L^{-j}_{Q\nu}\\
& - \frac{i}{\sqrt{2}}\epsilon^{\mu\nu}c\epsilon_{ij}L^{+i}_{Q\mu}\left(-\,\mathrm{d}\omega^{-j}_{Q\nu} + \frac{1}{2}((L^{-+}_{\nu} - L_{D\nu})\omega^{-j}_{Q} - L^{-j}_{Q\nu}(\omega^{-+} - \omega_D))\right)\\
& - \frac{i}{\sqrt{2}}\epsilon^{\mu\nu}c\epsilon_{ij}L^{+i}_{Q\mu}\left(-\tilde{L}^{j}{}_{k\nu}\omega^{-k}_{Q} + L^{-k}_{Q\nu}\tilde{\omega}^{j}{}_{k} - L^{-}_{P\nu}\omega^{+j}_{S} + L^{+j}_{S\nu}\omega^{-}_{P}\right)\\
& + \frac{i}{\sqrt{2}}\epsilon^{\mu\nu}c^*\epsilon^{ij}\left(-\,\mathrm{d}\omega^{+}_{Qi\mu} + \frac{1}{2}((-L^{-+}_{\mu} - L_{D\mu})\omega^{+}_{Qi} - L^{+}_{Qi\mu}(-\omega^{-+} - \omega_D))\right)L^{-}_{Qj\nu}\\
& + \frac{i}{\sqrt{2}}\epsilon^{\mu\nu}c^*\epsilon^{ij}\left(L^{k}{}_{i\mu}\omega^{+}_{Qk} - L^{+}_{Qk\mu}\omega^{k}{}_{i} + L^{+}_{P\mu}\omega^{-}_{Si} - L^{-}_{Si\mu}\omega^{+}_{P}\right)L^{-}_{Qj\nu}\\
& + \frac{i}{\sqrt{2}}\epsilon^{\mu\nu}c^*\epsilon^{ij}L^{+}_{Qi\mu}\left(-\,\mathrm{d}\omega^{-}_{Qj\nu} + \frac{1}{2}((L^{-+}_{\nu} - L_{D\nu})\omega^{-}_{Qj} - L^{-}_{Qj\nu}(\omega^{-+} - \omega_D))\right)\\
& + \frac{i}{\sqrt{2}}\epsilon^{\mu\nu}c^*\epsilon^{ij}L^{+}_{Qi\mu}\left(\tilde{L}^{k}{}_{j\nu}\omega^{-}_{Qk} - L^{-}_{Qk\nu}\tilde{\omega}^{k}{}_{j} + L^{-}_{P\nu}\omega^{+}_{Sj} - L^{+}_{Sj\nu}\omega^{-}_{P}\right)\\
=& \frac{i}{\sqrt{2}}\epsilon^{\mu\nu}\partial_\mu\left(c\epsilon_{ij}\left(\omega^{+i}_{Q}L^{-j}_{Q\nu} - L^{+i}_{Q\nu}\omega^{-j}_{Q}\right) - c^*\epsilon^{ij}\left(\omega^{+}_{Qi}L^{-}_{Qj\nu} - L^{+}_{Qi\nu}\omega^{-}_{Qj}\right)\right)\\
& - \frac{i}{\sqrt{2}}\epsilon^{\mu\nu}\left(c\epsilon_{ij}\left(\omega^{+i}_{Q}\partial_\mu L^{-j}_{Q\nu} - \partial_\mu L^{+i}_{Q\nu}\omega^{-j}_{Q}\right) - c^*\epsilon^{ij}\left(\omega^{+}_{Qi}\partial_\mu L^{-}_{Qj\nu} - \partial_\mu L^{+}_{Qi\nu}\omega^{-}_{Qj}\right)\right)\\
& - \frac{i}{2\sqrt{2}}\epsilon^{\mu\nu}c\epsilon_{ij}((-L^{-+}_{\mu} - L_{D\mu})\omega^{+i}_{Q} - L^{+i}_{Q\mu}(-\omega^{-+} - \omega_D))L^{-j}_{Q\nu}\\
& - \frac{i}{\sqrt{2}}\epsilon^{\mu\nu}c\epsilon_{ij}\left(-L^{i}{}_{k\mu}\omega^{+k}_{Q} + L^{+k}_{Q\mu}\omega^{i}{}_{k} - L^{+}_{P\mu}\omega^{-i}_{S} + L^{-i}_{S\mu}\omega^{+}_{P}\right)L^{-j}_{Q\nu}\\
& - \frac{i}{2\sqrt{2}}\epsilon^{\mu\nu}c\epsilon_{ij}L^{+i}_{Q\mu}((L^{-+}_{\nu} - L_{D\nu})\omega^{-j}_{Q} - L^{-j}_{Q\nu}(\omega^{-+} - \omega_D))\\
& - \frac{i}{\sqrt{2}}\epsilon^{\mu\nu}c\epsilon_{ij}L^{+i}_{Q\mu}\left(-\tilde{L}^{j}{}_{k\nu}\omega^{-k}_{Q} + L^{-k}_{Q\nu}\tilde{\omega}^{j}{}_{k} - L^{-}_{P\nu}\omega^{+j}_{S} + L^{+j}_{S\nu}\omega^{-}_{P}\right)\\
& + \frac{i}{2\sqrt{2}}\epsilon^{\mu\nu}c^*\epsilon^{ij}((-L^{-+}_{\mu} - L_{D\mu})\omega^{+}_{Qi} - L^{+}_{Qi\mu}(-\omega^{-+} - \omega_D))L^{-}_{Qj\nu}\\
& + \frac{i}{\sqrt{2}}\epsilon^{\mu\nu}c^*\epsilon^{ij}\left(L^{k}{}_{i\mu}\omega^{+}_{Qk} - L^{+}_{Qk\mu}\omega^{k}{}_{i} + L^{+}_{P\mu}\omega^{-}_{Si} - L^{-}_{Si\mu}\omega^{+}_{P}\right)L^{-}_{Qj\nu}\\
& + \frac{i}{2\sqrt{2}}\epsilon^{\mu\nu}c^*\epsilon^{ij}L^{+}_{Qi\mu}((L^{-+}_{\nu} - L_{D\nu})\omega^{-}_{Qj} - L^{-}_{Qj\nu}(\omega^{-+} - \omega_D))\\
& + \frac{i}{\sqrt{2}}\epsilon^{\mu\nu}c^*\epsilon^{ij}L^{+}_{Qi\mu}\left(\tilde{L}^{k}{}_{j\nu}\omega^{-}_{Qk} - L^{-}_{Qk\nu}\tilde{\omega}^{k}{}_{j} + L^{-}_{P\nu}\omega^{+}_{Sj} - L^{+}_{Sj\nu}\omega^{-}_{P}\right)
\end{aligned}
\tag{C.3.3b}
$$

The total derivatives can be neglected since they do not contribute to the action $S = \int \mathrm{d}^2\sigma\, \mathcal{L}$.

C.4 Equations of Motion

The equations of motion can be obtained from the requirement that the action is stationary, $\delta S = 0$ for any ω. It is convenient to write the variation of the Lagrangian as

$$
\begin{aligned}
\delta\mathcal{L} = {} & \text{total derivative} \\
& - \left(\sqrt{g}g^{\mu\nu}(\nabla_\mu L_{D\nu} + \hat{L}^a_\mu \hat{L}^a_\nu) + \frac{i}{\sqrt{2}}\epsilon^{\mu\nu}\left(c\epsilon_{ij}L^{+i}_{Q\mu}L^{-j}_{Q\nu} - c^*\epsilon^{ij}L^{+}_{Qi\mu}L^{-}_{Qj\nu}\right)\right)\omega_D \\
& + 0\omega^{-+} \\
& + \left(\frac{i}{2}\sqrt{g}g^{\mu\nu}\left(\left(\sigma^{A'}\right)^j{}_i\nabla_\mu + \left(\sigma^{A'}\right)^m{}_i L^j{}_{m\mu} - \left(\sigma^{A'}\right)^j{}_m L^m{}_{i\mu}\right)L^{A'}_\nu - \frac{i}{\sqrt{2}}\epsilon^{\mu\nu}\left(c\epsilon_{ik}L^{+j}_{Q\mu}L^{-k}_{Q\nu} + c^*\epsilon^{jk}L^{+}_{Qi\mu}L^{-}_{Qk\nu}\right)\right)\omega^i{}_j \\
& - \left(\frac{i}{2}\sqrt{g}g^{\mu\nu}\left(\left(\sigma^{A'}\right)^j{}_i\nabla_\mu + \left(\sigma^{A'}\right)^m{}_i \tilde{L}^j{}_{m\mu} - \left(\sigma^{A'}\right)^j{}_m \tilde{L}^m{}_{i\mu}\right)L^{A'}_\nu + \frac{i}{\sqrt{2}}\epsilon^{\mu\nu}\left(c\epsilon_{ki}L^{+k}_{Q\mu}L^{-j}_{Q\nu} + c^*\epsilon^{kj}L^{+}_{Qk\mu}L^{-}_{Qi\nu}\right)\right)\tilde{\omega}^i{}_j \\
& - \left(\sqrt{g}g^{\mu\nu}\left((\nabla_\mu - L_{D\mu} - L^{-+}_\mu)\hat{L}^-_\nu - L_{D\mu}L^+_{K\nu}\right) - \frac{i}{\sqrt{2}}\epsilon^{\mu\nu}\left(c\epsilon_{ij}L^{-i}_{Q\mu}L^{-j}_{S\nu} + c^*\epsilon^{ij}L^{-}_{Qi\mu}L^{-}_{Sj\nu}\right)\right)\omega^+_P \\
& - \left(\sqrt{g}g^{\mu\nu}\left((\nabla_\mu - L_{D\mu} + L^{-+}_\mu)\hat{L}^+_\nu - L_{D\mu}L^+_{K\nu}\right) + \frac{i}{\sqrt{2}}\epsilon^{\mu\nu}\left(c\epsilon_{ij}L^{+i}_{Q\mu}L^{+j}_{S\nu} + c^*\epsilon^{ij}L^{+}_{Qi\mu}L^{+}_{Sj\nu}\right)\right)\omega^-_P \\
& + \sqrt{g}g^{\mu\nu}\left(\frac{1}{2}(\nabla_\mu + L_{D\mu} - L^{-+}_\mu)\hat{L}^-_\nu - L_{D\mu}L^-_{P\nu}\right)\omega^+_K \\
& + \sqrt{g}g^{\mu\nu}\left(\frac{1}{2}(\nabla_\mu + L_{D\mu} + L^{-+}_\mu)\hat{L}^+_\nu - L_{D\mu}L^+_{P\nu}\right)\omega^-_K \\
& + \left(\sqrt{g}g^{\mu\nu}\left(\hat{L}^+_\mu L^-_{Qi\nu} + \frac{1}{2}L_{D\mu}L^+_{Si\nu} - \frac{i}{2}\left(\sigma^{A'}\right)^j{}_i L^{A'}_\nu L^+_{Sj\nu}\right) - \frac{i}{\sqrt{2}}\epsilon^{\mu\nu}c\left(\epsilon_{ij}\left(\partial_\mu + \frac{1}{2}(L^{-+}_\mu - L_{D\mu})\right) + \epsilon_{jk}\tilde{L}^k{}_{i\mu}\right)L^{+j}_{Q\nu}\right)\omega^{-i}_Q \\
& + \left(\sqrt{g}g^{\mu\nu}\left(\hat{L}^+_\mu L^{-i}_{Q\nu} - \frac{1}{2}L_{D\mu}L^{+i}_{S\nu} - \frac{i}{2}\left(\sigma^{A'}\right)^i{}_j L^{A'}_\nu L^{+j}_{S\nu}\right) + \frac{i}{\sqrt{2}}\epsilon^{\mu\nu}c^*\left(\epsilon^{ij}\left(\partial_\mu + \frac{1}{2}(L^{-+}_\mu - L_{D\mu})\right) - \epsilon^{jk}\tilde{L}^i{}_{k\mu}\right)L^+_{Qj\nu}\right)\omega^-_{Qi} \\
& - \left(\sqrt{g}g^{\mu\nu}\left(\hat{L}^-_\mu L^+_{Qi\nu} + \frac{1}{2}L_{D\mu}L^-_{Si\nu} - \frac{i}{2}\left(\sigma^{A'}\right)^j{}_i L^{A'}_\nu L^-_{Sj\nu}\right) - \frac{i}{\sqrt{2}}\epsilon^{\mu\nu}c\left(\epsilon_{ij}\left(\partial_\mu - \frac{1}{2}(L^{-+}_\mu + L_{D\mu})\right) - \epsilon_{kj}L^k{}_{i\mu}\right)L^{-j}_{Q\nu}\right)\omega^{+i}_Q \\
& - \left(\sqrt{g}g^{\mu\nu}\left(\hat{L}^-_\mu L^{+i}_{Q\nu} - \frac{1}{2}L_{D\mu}L^{-i}_{S\nu} + \frac{i}{2}\left(\sigma^{A'}\right)^i{}_j L^{A'}_\nu L^{-j}_{S\nu}\right) + \frac{i}{\sqrt{2}}\epsilon^{\mu\nu}c^*\left(\epsilon^{ij}\left(\partial_\mu - \frac{1}{2}(L^{-+}_\mu + L_{D\mu})\right) + \epsilon^{kj}L^i{}_{k\mu}\right)L^-_{Qj\nu}\right)\omega^+_{Qi} \\
& - \left(\frac{1}{2}\sqrt{g}g^{\mu\nu}\left(\hat{L}^+_\mu L^-_{Si\nu} - L_{D\mu}L^+_{Qi\nu} + i\left(\sigma^{A'}\right)^j{}_i L^{A'}_\nu L^+_{Qj\nu}\right) + \frac{i}{\sqrt{2}}\epsilon^{\mu\nu}c\epsilon_{ij}L^+_{P\mu}L^{-j}_{Q\nu}\right)\omega^{-i}_S \\
& - \left(\frac{1}{2}\sqrt{g}g^{\mu\nu}\left(\hat{L}^+_\mu L^{-i}_{S\nu} + L_{D\mu}L^{+i}_{Q\nu} + i\left(\sigma^{A'}\right)^i{}_j L^{A'}_\nu L^{+j}_{Q\nu}\right) + \frac{i}{\sqrt{2}}\epsilon^{\mu\nu}c^*\epsilon^{ij}L^+_{P\mu}L^-_{Qj\nu}\right)\omega^-_{Si} \\
& + \left(\frac{1}{2}\sqrt{g}g^{\mu\nu}\left(\hat{L}^-_\mu L^+_{Si\nu} - L_{D\mu}L^-_{Qi\nu} - i\left(\sigma^{A'}\right)^j{}_i L^{A'}_\nu L^-_{Qj\nu}\right) + \frac{i}{\sqrt{2}}\epsilon^{\mu\nu}c\epsilon_{ij}L^-_{P\mu}L^{+j}_{Q\nu}\right)\omega^{+i}_S \\
& + \left(\frac{1}{2}\sqrt{g}g^{\mu\nu}\left(\hat{L}^-_\mu L^{+i}_{S\nu} + L_{D\mu}L^{-i}_{Q\nu} - i\left(\sigma^{A'}\right)^i{}_j L^{A'}_\nu L^{-j}_{Q\nu}\right) + \frac{i}{\sqrt{2}}\epsilon^{\mu\nu}c^*\epsilon^{ij}L^-_{P\mu}L^+_{Qj\nu}\right)\omega^+_{Si}
\end{aligned}
\tag{C.4.1}
$$

We find that

$$0 = \sqrt{g}g^{\mu\nu}(\nabla_\mu L_{D\nu} + \hat{L}^a_\mu \hat{L}^a_\nu) + \frac{i}{\sqrt{2}}\epsilon^{\mu\nu}\left(c\epsilon_{ij}L^{+i}_{Q\mu}L^{-j}_{Q\nu} - c^*\epsilon^{ij}L^+_{Qi\mu}L^-_{Qj\nu}\right) \tag{C.4.2a}$$

$$0 = \frac{i}{2}\sqrt{g}g^{\mu\nu}\left(\left(\sigma^{A'}\right)^j{}_i\nabla_\mu + \left(\sigma^{A'}\right)^m{}_i L^j{}_{m\mu} - \left(\sigma^{A'}\right)^j{}_m L^m{}_{i\mu}\right)L^{A'}_\nu - \frac{i}{\sqrt{2}}\epsilon^{\mu\nu}\left(c\epsilon_{ik}L^{+j}_{Q\mu}L^{-k}_{Q\nu} + c^*\epsilon^{jk}L^+_{Qi\mu}L^-_{Qk\nu}\right) \tag{C.4.2b}$$

$$0 = \frac{i}{2}\sqrt{g}g^{\mu\nu}\left(\left(\sigma^{A'}\right)^j{}_i\nabla_\mu + \left(\sigma^{A'}\right)^m{}_i \tilde{L}^j{}_{m\mu} - \left(\sigma^{A'}\right)^j{}_m \tilde{L}^m{}_{i\mu}\right)L^{A'}_\nu + \frac{i}{\sqrt{2}}\epsilon^{\mu\nu}\left(c\epsilon_{ki}L^{+k}_{Q\mu}L^{-j}_{Q\nu} + c^*\epsilon^{kj}L^+_{Qk\mu}L^-_{Qi\nu}\right) \tag{C.4.2c}$$

$$0 = \sqrt{g}g^{\mu\nu}\left((\nabla_\mu - L_{D\mu} - L^{-+}_\mu)\hat{L}^-_\nu - L_{D\mu}L^+_{K\nu}\right) - \frac{i}{\sqrt{2}}\epsilon^{\mu\nu}\left(c\epsilon_{ij}L^{-i}_{Q\mu}L^{-j}_{S\nu} + c^*\epsilon^{ij}L^-_{Qi\mu}L^-_{Sj\nu}\right) \tag{C.4.2d}$$

$$0 = \sqrt{g}g^{\mu\nu}\left((\nabla_\mu - L_{D\mu} + L^{-+}_\mu)\hat{L}^+_\nu - L_{D\mu}L^+_{K\nu}\right) + \frac{i}{\sqrt{2}}\epsilon^{\mu\nu}\left(c\epsilon_{ij}L^{+i}_{Q\mu}L^{+j}_{S\nu} + c^*\epsilon^{ij}L^+_{Qi\mu}L^+_{Sj\nu}\right) \tag{C.4.2e}$$

$$0 = \sqrt{g}g^{\mu\nu}\left(\frac{1}{2}(\nabla_\mu + L_{D\mu} - L^{-+}_\mu)\hat{L}^-_\nu - L_{D\mu}L^-_{P\nu}\right) \tag{C.4.2f}$$

$$0 = \sqrt{g}g^{\mu\nu}\left(\frac{1}{2}(\nabla_\mu + L_{D\mu} + L^{-+}_\mu)\hat{L}^+_\nu - L_{D\mu}L^+_{P\nu}\right) \tag{C.4.2g}$$

$$0 = \sqrt{g}g^{\mu\nu}\left(\hat{L}^+_\mu L^-_{Qi\nu} + \frac{1}{2}L_{D\mu}L^+_{Si\nu} - \frac{i}{2}\left(\sigma^{A'}\right)^j{}_i L^{A'}_\nu L^+_{Sj\nu}\right) - \frac{i}{\sqrt{2}}\epsilon^{\mu\nu}c\left(\epsilon_{ij}\left(\partial_\mu + \frac{1}{2}(L^{-+}_\mu - L_{D\mu})\right) + \epsilon_{jk}\tilde{L}^k{}_{i\mu}\right)L^{+j}_{Q\nu} \tag{C.4.2h}$$

$$0 = \sqrt{g}g^{\mu\nu}\left(\hat{L}^+_\mu L^{-i}_{Q\nu} - \frac{1}{2}L_{D\mu}L^{+i}_{S\nu} - \frac{i}{2}\left(\sigma^{A'}\right)^i{}_j L^{A'}_\nu L^{+j}_{S\nu}\right) + \frac{i}{\sqrt{2}}\epsilon^{\mu\nu}c^*\left(\epsilon^{ij}\left(\partial_\mu + \frac{1}{2}(L^{-+}_\mu - L_{D\mu})\right) - \epsilon^{jk}\tilde{L}^i{}_{k\mu}\right)L^+_{Qj\nu} \tag{C.4.2i}$$

$$0 = \sqrt{g}g^{\mu\nu}\left(\hat{L}^-_\mu L^+_{Qi\nu} + \frac{1}{2}L_{D\mu}L^-_{Si\nu} - \frac{i}{2}\left(\sigma^{A'}\right)^j{}_i L^{A'}_\nu L^-_{Sj\nu}\right) - \frac{i}{\sqrt{2}}\epsilon^{\mu\nu}c\left(\epsilon_{ij}\left(\partial_\mu - \frac{1}{2}(L^{-+}_\mu + L_{D\mu})\right) - \epsilon_{kj}L^k{}_{i\mu}\right)L^{-j}_{Q\nu} \tag{C.4.2j}$$

$$0 = \sqrt{g}g^{\mu\nu}\left(\hat{L}^-_\mu L^{+i}_{Q\nu} - \frac{1}{2}L_{D\mu}L^{-i}_{S\nu} + \frac{i}{2}\left(\sigma^{A'}\right)^i{}_j L^{A'}_\nu L^{-j}_{S\nu}\right) + \frac{i}{\sqrt{2}}\epsilon^{\mu\nu}c^*\left(\epsilon^{ij}\left(\partial_\mu - \frac{1}{2}(L^{-+}_\mu + L_{D\mu})\right) + \epsilon^{kj}L^i{}_{k\mu}\right)L^-_{Qj\nu} \tag{C.4.2k}$$

$$0 = \frac{1}{2}\sqrt{g}g^{\mu\nu}\left(\hat{L}^+_\mu L^-_{Si\nu} - L_{D\mu}L^+_{Qi\nu} + i\left(\sigma^{A'}\right)^j{}_i L^{A'}_\nu L^+_{Qj\nu}\right) + \frac{i}{\sqrt{2}}\epsilon^{\mu\nu}c\epsilon_{ij}L^+_{P\mu}L^{-j}_{Q\nu} \tag{C.4.2l}$$

$$0 = \frac{1}{2}\sqrt{g}g^{\mu\nu}\left(\hat{L}^+_\mu L^{-i}_{S\nu} + L_{D\mu}L^{+i}_{Q\nu} + i\left(\sigma^{A'}\right)^i{}_j L^{A'}_\nu L^{+j}_{Q\nu}\right) + \frac{i}{\sqrt{2}}\epsilon^{\mu\nu}c^*\epsilon^{ij}L^+_{P\mu}L^-_{Qj\nu} \tag{C.4.2m}$$

$$0 = \frac{1}{2}\sqrt{g}g^{\mu\nu}\left(\hat{L}^-_\mu L^+_{Si\nu} - L_{D\mu}L^-_{Qi\nu} - i\left(\sigma^{A'}\right)^j{}_i L^{A'}_\nu L^-_{Qj\nu}\right) + \frac{i}{\sqrt{2}}\epsilon^{\mu\nu}c\epsilon_{ij}L^-_{P\mu}L^{+j}_{Q\nu} \tag{C.4.2n}$$

$$0 = \frac{1}{2}\sqrt{g}g^{\mu\nu}\left(\hat{L}^-_\mu L^{+i}_{S\nu} + L_{D\mu}L^{-i}_{Q\nu} - i\left(\sigma^{A'}\right)^i{}_j L^{A'}_\nu L^{-j}_{Q\nu}\right) + \frac{i}{\sqrt{2}}\epsilon^{\mu\nu}c^*\epsilon^{ij}L^-_{P\mu}L^+_{Qj\nu} \tag{C.4.2o}$$

D. MATHEMATICA PACKAGES

Two Mathematica packages have been written for and used in this thesis for algebraic simplification purposes.

D.1 Grassmann.m

The Grassmann package helps in dealing with Grassmann algebras. A Grassmann algebra with n generators $\theta_1, \ldots, \theta_n$ is spanned by the elements $1, \theta_i, \theta_i\theta_j, \ldots, \theta_1\theta_2 \ldots \theta_n$ with $\theta_i\theta_j = -\theta_j\theta_i$ and has dimension 2^n. Each element a can be written as a sum

$$a = \sum_{\epsilon \in \{0,1\}^n} a_\epsilon \theta_1^{\epsilon_1} \theta_2^{\epsilon_2} \ldots \theta_n^{\epsilon_n} \tag{D.1.1}$$

It is represented in Mathematica using the function `GA`, which makes use of the basis mentioned above. The argument of `GA` is a nested list, containing the coefficients a_ϵ according to the following scheme:

n	a
1	`GA[{`a_0`, `a_1`}]`
2	`GA[{{`a_{00}`, `a_{01}`}, {`a_{10}`, `a_{11}`}}]`
3	`GA[{{{`a_{000}`, `a_{001}`}, {`a_{010}`, `a_{011}`}}, {{`a_{100}`, `a_{101}`}, {`a_{110}`, `a_{111}`}}}]`
$\vdots$	$\vdots$

To avoid this cumbersome notation when entering Grassmann elements, the function `Theta` can be used. `Theta[n]` evaluates to the basis element θ_n. The Grassmann product between to elements a, b is entered as `a ** b` or `NonCommutativeMultiply[a, b]`. Grassmann elements are displayed in their basis expansion in canonical order. For example,

```
In[1] := 1 + Theta[1] + 2 Theta[2] ** Theta[1]
Out[1] := 1 + θ1 - 2 θ1 ** θ2
```

Powers of Grassmann elements can be entered as `a ^ n` or `Power[a, n]`. `Power[a, -1]` computes the inverse of a. Note that a is invertible iff $a_{0\ldots0} \neq 0$. There are two functions for division of Grassmann elements: `b / a` yields the right division ba^{-1}, while `a \ b` computes the left division $a^{-1}b$. Some other mathematical functions (exponential, logarithm, trigonometric and hyperbolic functions and their inverses) are defined via their power series expansion.

The left and right derivatives $\overrightarrow{\frac{\partial}{\partial\theta_n}} a$ and $a \overleftarrow{\frac{\partial}{\partial\theta_n}}$ are entered as `GADL[a, n]` and `GADR[a, n]`, respectively. Repeated derivatives are entered as

$$\ldots \overrightarrow{\frac{\partial}{\partial \theta_{n_2}}} \overrightarrow{\frac{\partial}{\partial \theta_{n_1}}} a \quad \leftrightarrow \quad \texttt{GADL[a, \{}n_1\texttt{, }n_2\texttt{, }\ldots\texttt{\}]}$$

$$a \overleftarrow{\frac{\partial}{\partial \theta_{n_1}}} \overleftarrow{\frac{\partial}{\partial \theta_{n_2}}} \ldots \quad \leftrightarrow \quad \texttt{GADR[a, \{}n_1\texttt{, }n_2\texttt{, }\ldots\texttt{\}]}$$

Note that the derivatives are applied in the same order as they are put inside the function call.

The package also supports handling Grassmann valued matrices. The ordinary dot product `Dot[M, N]` is replaced by `GADot[M, N]` and uses the Grassmann product. Matrix powers and inverses are computed using `GAMatrixPower[M, n]` and `GAInverse[M]`.

D.2 Clifford.m

The clifford algebra $Cl_{(p,q)}$ has $p+q$ generators $\Gamma_0, \ldots, \Gamma_{p+q-1}$, which satisfy

$$\Gamma_i \Gamma_j + \Gamma_j \Gamma_i = 2\eta_{ij} \tag{D.2.1}$$

where η is diagonal with the first p entries -1 and the last q entries +1. Explicitly, one finds

$$\Gamma_i \Gamma_j = \begin{cases} -\Gamma_j \Gamma_i & \text{if } i \neq j \\ -1 & \text{if } i = j < p \\ 1 & \text{if } i = j \geq p \end{cases} \tag{D.2.2}$$

A common basis is given by the 2^{p+q} elements $1, \Gamma_i, \Gamma_{ij} := \Gamma_i \Gamma_j, \ldots, \Gamma_{01\ldots} := \Gamma_0 \Gamma_1 \ldots \Gamma_{p+q-1}$. Any Clifford element c can thus be expressed as

$$c = \sum_{\epsilon \in \{0,1\}^{p+q}} c_\epsilon \Gamma_0^{\epsilon_0} \Gamma_1^{\epsilon_1} \ldots \Gamma_{p+q-1}^{\epsilon_{p+q-1}} \tag{D.2.3}$$

In Mathematica, Clifford elements are expressed using the function `CA`. The element $c \in Cl_{(p,q)}$ is represented by

`CA[p, q, {...{{{`$c_{00\ldots00}$`, `$c_{00\ldots01}$`}, {`$c_{00\ldots10}$`, `$c_{00\ldots11}$`}}, ...}...}]`

The basis element $\Gamma_{i_1 i_2 \ldots i_n}$ can be entered as `Cliff[p, q, {`i_1`, `i_2`, ..., `i_n`}]`. The Clifford product of two elements a, b is computed by entering `a ** b` or `NonCommutativeMultiply[a, b]`. Clifford elements are always displayed in their standard basis expansion. For example,

```
In[1] := (Cliff[0, 2, {1}] + Cliff[0, 2, {2}]) ** Cliff[0, 2, {1}]
Out[1] := 1 - Γ_{1,2}
```

Any Clifford algebra is equipped with a set of (anti)automorphisms, which are also implemented in the Mathematica package. The following table lists their usage.

(Anti)automorphism	Notation	Mathematica function
Involution	$\alpha(x)$	`CAAlpha[x]`
Transpose	x^t	`CATranspose[x]`
Conjugate	$\bar{x}$	`CAConjugate[x]`
Adjoint	$x^\dagger$	`CAAdjoint[x]`
Inverse	x^{-1}	`CAInverse[x]`

The inverse can also be entered as `Power[x, -1]`. Powers are entered as `Power[x, n]`. The left and right division $a^{-1}b$ and ba^{-1} are written as `a \ b` and `b / a`, respectively.

D.3 Listings

D.3.1 Grassmann.m

```
(*
    Grassmann.m

    Package Algebra`Grassmann`

    Functions for dealing with Grassmann-Algebras
*)

BeginPackage["Algebra`Grassmann`"];

GA::usage = "GA[elements] represents a Grassmann number using the standard expansion
    \!\(\[Theta] = a + a\_1\[Theta]\_1 + a\_2\[Theta]\_2 + a\_\(12\)\[Theta]\_1 **
    \[Theta]\_2 + \[Ellipsis]\)";

Theta::usage = "Theta[i] represents the Grassmann basis element \(\[Theta]\_i\).";

\[Theta]::usage = "Grassmann basis element \(\[Theta]\_i\)";

GADL::usage = "GADL[f, n] is the left derivative \!\(\(\[PartialD] \/
    \(\[PartialD] \[Theta]\_n\)\) \& \[RightVector] f\)\n
    \!\(GADL[f, {n\_1, n\_2, \[Ellipsis]}]\) is the repeated left derivative
    \!\(\[Ellipsis] \(\[PartialD] \/ \(\[PartialD] \[Theta]\_\(n\_2\)\)\) \&
    \[RightVector]  \(\[PartialD] \/ \(\[PartialD] \[Theta]\_\(n\_1\)\)\) \&
    \[RightVector] f\)"

GADR::usage = "GADR[f, n] is the right derivative \!\(f \(\[PartialD] \/
    \(\[PartialD] \[Theta]\_n\)\) \& \[LeftVector]\)\n
    \!\(GADR[f, {n\_1, n\_2, \[Ellipsis]}]\) is the repeated right derivative
    \!\(f \(\[PartialD] \/ \(\[PartialD] \[Theta]\_\(n\_1\)\)\) \& \[LeftVector]
    \(\[PartialD] \/ \(\[PartialD] \[Theta]\_\(n\_2\)\)\) \& \[LeftVector]
    \[Ellipsis]\)"

GADot::usage = "\!\(GADot[m\_1, m\_2, \[Ellipsis]]\) computes the dot product of
    Grassmann valued matrizes."

GAMatrixPower::usage = "GAMatrixPower[m, n] computes the n'th matrix power
    \!\(m\^n\) of a Grassmann valued matrix m."

GAInverse::usage = "GAInverse[m] computes the inverse of a Grassmann valued matrix m."

Begin["`Private`"];

(* A TensorRank[] that counts only lists. *)

garank[x_] := If[Head[x] === List, TensorRank[x], 0];
```

```
(* Test whether x is a full tensor with two elements at each level.*)

gatest[x_] := Or[Head[x] =!= List, And[Length[x] == 2,
    garank[x[[1]]] == garank[x[[2]]],   gatest[x[[1]]], gatest[x[[2]]]]];

(* Generate a full tensor of rank n with zero entries. *)

zero[n_Integer /; n >= 0] := Table @@ Join[{0}, Table[{2}, {n}]];

(* The n'th Grassmann basis element. *)

Theta[n_Integer /; n > 0] := Module[{t}, t = zero[n];
    Part[t, Sequence @@ Table[1, {n - 1}], 2] = 1; GA[t]];

(* Append one level with zeros to a full tensor. *)

expand1[x_ /; gatest[x]] := If[Head[x] === List, Map[{#, 0}&, x, {garank[x]}], {x, 0}];

(* Append the number of levels that are necessary to form a tensor of rank n. *)

expand[x_ /; gatest[x], n_Integer] := If[n > garank[x],
    Nest[expand1, x, n - garank[x]], x];

(* Remove one level of zeros. *)

reduce1[x_ /; gatest[x]] := Map[#[[1]]&, x, {garank[x] - 1}];

(* Remove all levels that contain only zeros. *)

reduce[x_ /; gatest[x]] := Module[{n}, If[Head[x] === List, n = garank[x];
    If[And @@ ((# === 0)& /@ Flatten[{Map[#[[2]]&, x, {n - 1}]}]),
    reduce[reduce1[x]], x], x]];

(* Replace all Grassmann elements by their first element. *)

numpart[x_] := x /. GA -> (Part[#, Sequence @@ Table[1, {garank[#]}]]&);

(* The sum is computed for each component. *)

GA /: GA[x_ /; gatest[x]] + GA[y_ /; gatest[y]] := Module[{n},
    n = Max[garank[x], garank[y]];
    GA[reduce[expand[x, n] + expand[y, n]]]];

(* This allows adding an ordinary number to a Grassmann element. *)

GA /: GA[x_ /; gatest[x]] + y_ /; And[Head[y] =!= List, Head[y] =!= GA] :=
    GA[x] + GA[y];
```

(* The product formula uses the fact that $\theta_i\theta_j = -\theta_j\theta_i$. *)

```
GA /: GA[x_ /; gatest[x]] ** GA[y_ /; gatest[y]] := Module[{n, pl, x2, y2, t, i, j},
    n = Max[garank[x], garank[y]];  x2 = expand[x, n]; y2 = expand[y, n];
    pl = Position[x2, _, {n}, Heads -> False]; t = zero[n];
    Do[If[And @@ ((# < 4)& /@ (pl[[i]] + pl[[j]])),
    Part[t, Sequence @@ (pl[[i]] + pl[[j]] - 1)] +=
    Signature[Join[Position[pl[[i]], 2], Position[pl[[j]], 2]]] *
    Part[x2, Sequence @@ pl[[i]]] * Part[y2, Sequence @@ pl[[j]]]],
    {i, 1, Length[pl]}, {j, 1, Length[pl]}];
    GA[reduce[t]]];

(* The following three definitions allow multiplication with ordinary numbers. *)

GA /: GA[x_ /; gatest[x]] y_ /; And[Head[y] =!= List, Head[y] =!= GA] := GA[x y];

GA /: GA[x_ /; gatest[x]] ** y_ /; And[Head[y] =!= List, Head[y] =!= GA] := GA[x y];

GA /: y_ ** GA[x_ /; gatest[x]] /; And[Head[y] =!= List, Head[y] =!= GA] := GA[x y];

(* We also give a definition for the Grassmann product if both factors are numbers. *)

Unprotect[NonCommutativeMultiply];

NonCommutativeMultiply[x_ /; And[Head[x] =!= List, Head[x] =!= GA,
    !MatchQ[x, Subscript[\[Theta], _]]],
    y_ /; And[Head[y] =!= List, Head[y] =!= GA,
    !MatchQ[y, Subscript[\[Theta], _]]]] := GA[x y];

Protect[NonCommutativeMultiply];
```

(* The inversion formula uses the power series expansion of $\frac{1}{1+y}$,
where $y = \frac{x-a_0}{a_0}$ is a pure Grassmann element and
$x^{-1} = \frac{1}{a_0}\frac{1}{1+y}$. The series expansion converges since
every power series of pure Grassmann elements ins finite. *)

```
inverse[GA[x_ /; gatest[x]]] := Module[{x1, x2, n, i},
    If[Head[x] =!= List, GA[1 / x], n = garank[x];
    x1 = Part[x, Sequence @@ Table[1, {n}]]; x2 = x / x1;
    Part[x2, Sequence @@ Table[1, {n}]] = 0;
    Sum[Power[-1, i] Power[GA[x2], i], {i, 0, n}] / x1]];

(* Powers are computed recursively. *)

GA /: Power[GA[x_ /; gatest[x]], n_Integer /; n > 1] := Power[GA[x], n - 1] ** GA[x];

GA /: Power[GA[x_ /; gatest[x]], 1] := GA[x];
```

```
GA /: Power[GA[x_ /; gatest[x]], 0] := GA[1];

GA /: Power[GA[x_ /; gatest[x]], n_Integer /; n < 0] := Power[inverse[GA[x]], -n];

(* Right division: x / y = xy^-1 *)

GA /: Divide[GA[x_ /; gatest[x]], GA[y_ /; gatest[y]]] := GA[x] ** inverse[GA[y]];

(* These two formulas allow right division with ordinary numbers. *)

GA /: Divide[GA[x_ /; gatest[x]], y_ /; And[Head[y] =!= List, Head[y] =!= GA]] :=
    GA[x / y];

GA /: Divide[x_ /; And[Head[x] =!= List, Head[x] =!= GA], GA[y_ /; gatest[y]]] :=
    GA[x] ** inverse[GA[y]];

(* Left division: x \ y = x^-1 y *)

GA /: Backslash[GA[x_ /; gatest[x]], GA[y_ /; gatest[y]]] := inverse[GA[x]] ** GA[y];

(* These two formulas allow left division with ordinary numbers. *)

GA /: Backslash[GA[x_ /; gatest[x]], y_ /; And[Head[y] =!= List, Head[y] =!= GA]] :=
    inverse[GA[x]] ** GA[y];

GA /: Backslash[x_ /; And[Head[x] =!= List, Head[x] =!= GA], GA[y_ /; gatest[y]]] :=
    GA[y / x];

(* Several numerical functions are implemented using their power series. *)

GA /: Exp[GA[x_ /; gatest[x]]] := Module[{x1, x2, n, i},
    If[Head[x] =!= List, GA[Exp[x]], n = garank[x];
    x1 = Part[x, Sequence @@ Table[1, {n}]]; x2 = x;
    Part[x2, Sequence @@ Table[1, {n}]] = 0;
    Exp[x1] Sum[Power[GA[x2], i] / i!, {i, 0, n}]]];

GA /: Log[GA[x_ /; gatest[x]]] := Module[{x1, x2, n, i},
    If[Head[x] =!= List, GA[Log[x]], n = garank[x];
    x1 = Part[x, Sequence @@ Table[1, {n}]]; x2 = -x / x1;
    Part[x2, Sequence @@ Table[1, {n}]] = 0;
    1 - Sum[Power[GA[x2], i] / i!, {i, 0, n}] + Log[x1]]];

GA /: Cos[GA[x_ /; gatest[x]]] := Module[{x1, x2, n, i},
    If[Head[x] =!= List, GA[Cos[x]], n = garank[x];
    x1 = Part[x, Sequence @@ Table[1, {n}]]; x2 = x;
    Part[x2, Sequence @@ Table[1, {n}]] = 0;
    Cos[x1] Sum[Power[-1, i] Power[GA[x2], 2i] / (2i)!, {i, 0, n / 2}] -
    Sin[x1] Sum[Power[-1, i] Power[GA[x2], 2i + 1] / (2i + 1)!, {i, 0, n / 2}]]];
```

```
GA /: Sin[GA[x_ /; gatest[x]]] := Module[{x1, x2, n, i},
    If[Head[x] =!= List, GA[Sin[x]], n = garank[x];
    x1 = Part[x, Sequence @@ Table[1, {n}]]; x2 = x;
    Part[x2, Sequence @@ Table[1, {n}]] = 0;
    Sin[x1] Sum[Power[-1, i] Power[GA[x2], 2i] / (2i)!, {i, 0, n / 2}] +
    Cos[x1] Sum[Power[-1, i] Power[GA[x2], 2i + 1] / (2i + 1)!, {i, 0, n / 2}]]];

GA /: Sec[GA[x_ /; gatest[x]]] := inverse[Cos[GA[x]]];

GA /: Csc[GA[x_ /; gatest[x]]] := inverse[Sin[GA[x]]];

GA /: Tan[GA[x_ /; gatest[x]]] := Sin[GA[x]] ** inverse[Cos[GA[x]]];

GA /: Cot[GA[x_ /; gatest[x]]] := Cos[GA[x]] ** inverse[Sin[GA[x]]];

GA /: Cosh[GA[x_ /; gatest[x]]] := Module[{x1, x2, n, i},
    If[Head[x] =!= List, GA[Cosh[x]], n = garank[x];
    x1 = Part[x, Sequence @@ Table[1, {n}]]; x2 = x;
    Part[x2, Sequence @@ Table[1, {n}]] = 0;
    Cosh[x1] Sum[Power[GA[x2], 2i] / (2i)!, {i, 0, n / 2}] +
    Sinh[x1] Sum[Power[GA[x2], 2i + 1] / (2i + 1)!, {i, 0, n / 2}]]];

GA /: Sinh[GA[x_ /; gatest[x]]] := Module[{x1, x2, n, i},
    If[Head[x] =!= List, GA[Sinh[x]], n = garank[x];
    x1 = Part[x, Sequence @@ Table[1, {n}]]; x2 = x;
    Part[x2, Sequence @@ Table[1, {n}]] = 0;
    Sinh[x1] Sum[Power[GA[x2], 2i] / (2i)!, {i, 0, n / 2}] +
    Cosh[x1] Sum[Power[GA[x2], 2i + 1] / (2i + 1)!, {i, 0, n / 2}]]];

GA /: Sech[GA[x_ /; gatest[x]]] := inverse[Cosh[GA[x]]];

GA /: Csch[GA[x_ /; gatest[x]]] := inverse[Sinh[GA[x]]];

GA /: Tanh[GA[x_ /; gatest[x]]] := Sinh[GA[x]] ** inverse[Cosh[GA[x]]];

GA /: Coth[GA[x_ /; gatest[x]]] := Cosh[GA[x]] ** inverse[Sinh[GA[x]]];

GA /: ArcSin[GA[x_ /; gatest[x]]] := Module[{x1, x2, n, i},
    If[Head[x] =!= List, GA[ArcSin[x]], n = garank[x];
    x1 = Part[x, Sequence @@ Table[1, {n}]]; x2 = x;
    Part[x2, Sequence @@ Table[1, {n}]] = 0;
    Sum[Derivative[i][ArcSin][x1] Power[GA[x2], i] / i!, {i, 0, n}]]];

GA /: ArcCos[GA[x_ /; gatest[x]]] := Module[{x1, x2, n, i},
    If[Head[x] =!= List, GA[ArcCos[x]], n = garank[x];
    x1 = Part[x, Sequence @@ Table[1, {n}]]; x2 = x;
    Part[x2, Sequence @@ Table[1, {n}]] = 0;
    Sum[Derivative[i][ArcCos][x1] Power[GA[x2], i] / i!, {i, 0, n}]]];
```

```
GA /: ArcTan[GA[x_ /; gatest[x]]] := Module[{x1, x2, n, i},
    If[Head[x] =!= List, GA[ArcTan[x]], n = garank[x];
    x1 = Part[x, Sequence @@ Table[1, {n}]]; x2 = x;
    Part[x2, Sequence @@ Table[1, {n}]] = 0;
    Sum[Derivative[i][ArcTan][x1] Power[GA[x2], i] / i!, {i, 0, n}]]];

GA /: ArcSec[GA[x_ /; gatest[x]]] := Module[{x1, x2, n, i},
    If[Head[x] =!= List, GA[ArcSec[x]], n = garank[x];
    x1 = Part[x, Sequence @@ Table[1, {n}]]; x2 = x;
    Part[x2, Sequence @@ Table[1, {n}]] = 0;
    Sum[Derivative[i][ArcSec][x1] Power[GA[x2], i] / i!, {i, 0, n}]]];

GA /: ArcCsc[GA[x_ /; gatest[x]]] := Module[{x1, x2, n, i},
    If[Head[x] =!= List, GA[ArcCsc[x]], n = garank[x];
    x1 = Part[x, Sequence @@ Table[1, {n}]]; x2 = x;
    Part[x2, Sequence @@ Table[1, {n}]] = 0;
    Sum[Derivative[i][ArcCsc][x1] Power[GA[x2], i] / i!, {i, 0, n}]]];

GA /: ArcCot[GA[x_ /; gatest[x]]] := Module[{x1, x2, n, i},
    If[Head[x] =!= List, GA[ArcCot[x]], n = garank[x];
    x1 = Part[x, Sequence @@ Table[1, {n}]]; x2 = x;
    Part[x2, Sequence @@ Table[1, {n}]] = 0;
    Sum[Derivative[i][ArcCot][x1] Power[GA[x2], i] / i!, {i, 0, n}]]];

GA /: ArcSinh[GA[x_ /; gatest[x]]] := Module[{x1, x2, n, i},
    If[Head[x] =!= List, GA[ArcSinh[x]], n = garank[x];
    x1 = Part[x, Sequence @@ Table[1, {n}]]; x2 = x;
    Part[x2, Sequence @@ Table[1, {n}]] = 0;
    Sum[Derivative[i][ArcSinh][x1] Power[GA[x2], i] / i!, {i, 0, n}]]];

GA /: ArcCosh[GA[x_ /; gatest[x]]] := Module[{x1, x2, n, i},
    If[Head[x] =!= List, GA[ArcCosh[x]], n = garank[x];
    x1 = Part[x, Sequence @@ Table[1, {n}]]; x2 = x;
    Part[x2, Sequence @@ Table[1, {n}]] = 0;
    Sum[Derivative[i][ArcCosh][x1] Power[GA[x2], i] / i!, {i, 0, n}]]];

GA /: ArcTanh[GA[x_ /; gatest[x]]] := Module[{x1, x2, n, i},
    If[Head[x] =!= List, GA[ArcTanh[x]], n = garank[x];
    x1 = Part[x, Sequence @@ Table[1, {n}]]; x2 = x;
    Part[x2, Sequence @@ Table[1, {n}]] = 0;
    Sum[Derivative[i][ArcTanh][x1] Power[GA[x2], i] / i!, {i, 0, n}]]];

GA /: ArcSech[GA[x_ /; gatest[x]]] := Module[{x1, x2, n, i},
    If[Head[x] =!= List, GA[ArcSech[x]], n = garank[x];
    x1 = Part[x, Sequence @@ Table[1, {n}]]; x2 = x;
    Part[x2, Sequence @@ Table[1, {n}]] = 0;
    Sum[Derivative[i][ArcSech][x1] Power[GA[x2], i] / i!, {i, 0, n}]]];

GA /: ArcCsch[GA[x_ /; gatest[x]]] := Module[{x1, x2, n, i},
```

```
    If[Head[x] =!= List, GA[ArcCsch[x]], n = garank[x];
    x1 = Part[x, Sequence @@ Table[1, {n}]]; x2 = x;
    Part[x2, Sequence @@ Table[1, {n}]] = 0;
    Sum[Derivative[i][ArcCsch][x1] Power[GA[x2], i] / i!, {i, 0, n}]]];

GA /: ArcCoth[GA[x_ /; gatest[x]]] := Module[{x1, x2, n, i},
    If[Head[x] =!= List, GA[ArcCoth[x]], n = garank[x];
    x1 = Part[x, Sequence @@ Table[1, {n}]]; x2 = x;
    Part[x2, Sequence @@ Table[1, {n}]] = 0;
    Sum[Derivative[i][ArcCoth][x1] Power[GA[x2], i] / i!, {i, 0, n}]]];

GA /: fkt_[GA[x_ /; gatest[x]]] := Module[{x1, x2, n, i},
    If[Head[x] =!= List, GA[fkt[x]], n = garank[x];
    x1 = Part[x, Sequence @@ Table[1, {n}]]; x2 = x;
    Part[x2, Sequence @@ Table[1, {n}]] = 0;
    Sum[Derivative[i][fkt][x1] Power[GA[x2], i] / i!, {i, 0, n}]]] /;
    MemberQ[Attributes[fkt], NumericFunction];

(* The derivative of GA is the identity. *)

Unprotect[Derivative];
Derivative[x_ /; gatest[x]][GA][y_ /; gatest[y]] := GA[reduce[x]];
Protect[Derivative];
```

(* The left derivative $\overrightarrow{\frac{\partial}{\partial \theta_m}}$. *)

```
GADL[GA[x_ /; gatest[x]], m_Integer /; m > 0] := Module[{pl, t, i, n},
    n = garank[x]; If[m > n, GA[0], pl = Position[x, _, {n}, Heads -> False];
    t = zero[n]; Do[If[pl[[i,m]] == 1,
    Part[t, Sequence @@ pl[[i]]] = Power[-1, Plus @@ Take[pl[[i]] - 1, m - 1]] *
    Part[x, Sequence @@ Join[Take[pl[[i]], m - 1], {2}, Take[pl[[i]], m - n]]]],
    {i, 1, Length[pl]}]; GA[reduce[t]]]];

(* The zero'th left derivative is the identity. *)

GADL[GA[x_ /; gatest[x]], {}] := GA[x];

(* Left derivatives may be nested. *)

GADL[GA[x_ /; gatest[x]], {m_Integer /; m > 0, mm___Integer}] :=
    GADL[GADL[GA[x], m], {mm}];

(* Left derivatives of ordinary numbers vanish. *)

GADL[x_ /; Head[x] =!= GA, _] := 0
```

(* The right derivative $\overleftarrow{\frac{\partial}{\partial \theta_m}}$. *)

```
GADR[GA[x_ /; gatest[x]], m_Integer /; m > 0] := Module[{pl, t, i, n},
    n = garank[x]; If[m > n, GA[0], pl = Position[x, _, {n}, Heads -> False];
    t = zero[n]; Do[If[pl[[i,m]] == 1,
    Part[t, Sequence @@ pl[[i]]] = Power[-1, Plus @@ Take[pl[[i]] - 1, m - n]] *
    Part[x, Sequence @@ Join[Take[pl[[i]], m - 1], {2}, Take[pl[[i]], m - n]]]],
    {i, 1, Length[pl]}]; GA[reduce[t]]]];

(* The zero'th right derivative is the identity. *)

GADR[GA[x_ /; gatest[x]], {}] := GA[x];

(* Right derivatives may be nested. *)

GADR[GA[x_ /; gatest[x]], {m_Integer /; m > 0, mm___Integer}] :=
    GADR[GADR[GA[x], m], {mm}];

(* Right derivatives of ordinary numbers vanish. *)

GADR[x_ /; Head[x] =!= GA, _] := 0

(* The product of only one Grassmann matrix is the identity. *)

GADot[x_] := x;

(* The product of two Grassmann matrizes, written as an inner product. *)

GADot[x_, y_] := Inner[NonCommutativeMultiply, x, y];

(* Products of more than one Grassmann matrix are defined recursively. *)

GADot[x_, y_, z__] := GADot[Inner[NonCommutativeMultiply, x, y], z];

(* The inverse of a Grassmann matrix. *)

GAInverse::matrix = "`1` is not a matrix."

GAInverse::matsq = "`1` is not a square matrix."

GAInverse::sing = "Matrix `1` is singular."

(* We use a speciel pivoting algorithm to compute the inverse. *)

GAInverse[m_] := Module[{n, res, num, i, j, k, max, mr, mc, ar, ac, mat, ip, mul},
    If[!MatrixQ[m], Return[Message[GAInverse::matrix, m]]];
    If[Length[Union[Dimensions[m]]] > 1, Return[Message[GAInverse::matsq, m]]];
    num = numpart[m];   If[Det[num] == 0, Return[Message[GAInverse::sing, m]]];
    n = Dimensions[m][[1]]; mat = m;    res = Map[GA, IdentityMatrix[n], {2}];
    Do[max = 0; mr = i; mc = i; Do[If[Abs[num[[j,k]]] > max, max = Abs[num[[j,k]]];
    mr = j; mc = k], {j, i, n}, {k, i, i}];
```

```
    ar = Range[n]; ac = Range[n]; ar[[i]] = mr; ar[[mr]] = i; ac[[i]] = mc;
    ac[[mc]] = i;   res = res[[ar,ac]]; mat = mat[[ar,ac]]; num = num[[ar,ac]];
    ip = inverse[mat[[i,i]]];   Do[mat[[i,j]] = ip ** mat[[i,j]];
    res[[i,j]] = ip ** res[[i,j]], {j, 1, n}]; num[[i]] /= num[[i,i]];
    Do[ip = mat[[j,i]]; Do[res[[j,k]] = res[[j,k]] - ip ** res[[i,k]];
    mat[[j,k]] = mat[[j,k]] - ip ** mat[[i,k]], {k, 1, n}];
    num[[j]] -= num[[j,i]] num[[i]], {j, i + 1, n}],
    {i, 1, n}]; Do[res[[j,k]] = res[[j,k]] - mat[[j,i]] ** res[[i,k]],
    {i, n, 2, -1}, {j, i - 1, 1, -1}, {k, 1, n}]; res];

(* Matrix powers are computed as repeated products. *)

GAMatrixPower[m_, n_Integer /; n > 0] := GADot @@ Table[m, {n}];

(* The zero'th matrix power is the identity. *)

GAMatrixPower[m_, 0] := IdentityMatrix[Length[m]];

(* Negative matrix powers use the inverse matrix. *)

GAMatrixPower[m_, n_Integer /; n < 0] := GADot @@ Table[GAInverse[m], {-n}];

(* Finally, all Grassmann numbers are printed in the standard basis. *)

Format[GA[x_ /; gatest[x]]] := Module[{pl, i, j, p, n}, n = garank[x];
    pl = Position[x, _, {n}, Heads -> False];
    p = Table[1, {Length[pl]}]; Do[Do[If[pl[[i,j]] == 2,
    If[p[[i]] === 1, p[[i]] = Subscript[\[Theta], j],
    p[[i]] = p[[i]] ** Subscript[\[Theta], j]]], {j, 1, n}];
    p[[i]] = Part[x, Sequence @@ pl[[i]]] p[[i]], {i, 1, Length[pl]}]; Plus @@ p];

End[];
EndPackage[];
```

D.3.2 Clifford.m

```
(*
    Clifford.m

    Package Algebra`Clifford`

    Functions for dealing with Clifford-Algebras
*)

BeginPackage["Algebra`Clifford`"];

CA::usage = "CA[p, q, elements] represents an element of the Clifford algebra
    \!\(Cl\_\(p,q\)\).\)";
```

```
Cliff::usage = "Cliff[p, q, {i, j, \[Ellipsis]}] represents the Clifford basis
    element \!\(\[CapitalGamma]\_\(i, j, \[Ellipsis]\) \[Element] Cl\_\(p,q\)\).";

\[CapitalGamma]::usage = "Clifford basis element
    \!\(\[CapitalGamma]\_\(i, j, \[Ellipsis]\)\)";

CAAlpha::usage = "CAAlpha[x] computes the involution
    \!\(\[Alpha](x)\) of a Clifford element x.";

CATranspose::usage = "CATranspose[x] computes the transpose
    \!\(x\^t\) of a Clifford element x.";

CAConjugate::usage = "CAConjugate[x] computes the conjugate
    \!\(x\&_\) of a Clifford element x.";

CAAdjoint::usage = "CAAdjoint[x] computes the adjoint
    \!\(x\^\[Dagger]\) of a Clifford element x.";

CAInverse::usage = "CAInverse[x] computes the inverse
    \!\(x\^\(-1\)\) of a Clifford element x.";

Begin["`Private`"];

(* A TensorRank[] that counts only lists. *)

carank[x_] := If[Head[x] === List, TensorRank[x], 0];

(* Test whether x is a full tensor with two elements at each level.*)

catest[x_] := Or[Head[x] =!= List, And[Length[x] == 2,
    carank[x[[1]]] == carank[x[[2]]],   catest[x[[1]]], catest[x[[2]]]]];

(* Test whether a Clifford element has the correct syntax. *)

catest2[x_] := MatchQ[x, CA[p_Integer, q_Integer, e_] /;
    And[catest[e], carank[e] == p + q]];

(* Generate a full tensor of rank n with zero entries. *)

zero[n_Integer /; n >= 0] := Table @@ Join[{0}, Table[{2}, {n}]];

(* Generate a full tensor of rank n with only the first element 1. *)

one[n_Integer /; n >= 0] := ReplacePart[zero[n], 1, Table[1, {n}]];

(* Replace all Clifford elements by their first element. *)

numpart[x_] := x /. CA -> (Part[#3, Sequence @@ Table[1, {carank[#3]}]]&);
```

```
(* Helper function for computing the involution. *)

alpha[x_ /; catest[x]] := If[carank[x] == 0, x,
    Array[Power[-1, Plus @@ (List[##] - 1)]&, Table[2, {carank[x]}]] x];

(* Helper function for computing the traspose. *)

transpose[x_ /; catest[x]] := If[carank[x] == 0, x,
    Array[Power[-1, (Plus @@ (List[##] - 1)) ((Plus @@ (List[##] - 1)) - 1) / 2]&,
    Table[2, {carank[x]}]] x];

(* The product of two ordinary numbers is just the usual product. *)

prod[0, 0, x_ /; catest[x], y_ /; catest[y]] := x y /; 0 == carank[x] == carank[y];

(* If the lowest $\Gamma_i$ satisfies $\Gamma_i^2 = 1$, use the formula
    $(a + \Gamma_i b)(c + \Gamma_i d) = (ac + \alpha(b)d) + \Gamma_i(bc + \alpha(a)d)$. *)

prod[0, q_Integer /; q > 0, x_ /; catest[x], y_ /; catest[y]] :=
    {prod[0, q - 1, x[[1]], y[[1]]] + prod[0, q - 1, alpha[x[[2]]], y[[2]]],
    prod[0, q - 1, alpha[x[[1]]], y[[2]]] + prod[0, q - 1, x[[2]], y[[1]]]} /;
    q == carank[x] == carank[y];

(* If the lowest $\Gamma_i$ satisfies $\Gamma_i^2 = -1$, use the formula
    $(a + \Gamma_i b)(c + \Gamma_i d) = (ac - \alpha(b)d) + \Gamma_i(bc + \alpha(a)d)$. *)

prod[p_Integer /; p > 0, q_Integer /; q >= 0, x_ /; catest[x], y_ /; catest[y]] :=
    {prod[p - 1, q, x[[1]], y[[1]]] - prod[p - 1, q, alpha[x[[2]]], y[[2]]],
    prod[p - 1, q, alpha[x[[1]]], y[[2]]] + prod[p - 1, q, x[[2]], y[[1]]]} /;
    p + q == carank[x] == carank[y];

(* The Clifford basis elements. *)

Cliff[p_Integer /; p >= 0, q_Integer /; q >= 0, x_List] := Module[{tab, ind},
    tab = zero[p + q]; ind = Array[If[MemberQ[x, #], 2, 1]&, {p + q}];
    CA[p, q, ReplacePart[tab, Signature[x], ind]]];

(* The sum is computed for each component. *)

CA /: CA[p_Integer /; p >= 0, q_Integer /; q >= 0, x_ /; catest[x]] +
    CA[p2_Integer /; p2 >= 0, q2_Integer /; q2 >= 0, y_ /; catest[y]] :=
    CA[p, q, x + y] /; And[p + q == carank[x] == carank[y], p == p2, q == q2];

(* This allows adding an ordinary number to a Clifford element. *)

CA /: CA[p_Integer /; p >= 0, q_Integer /; q >= 0, x_ /; catest[x]] +
    y_ /; And[Head[y] =!= List, Head[y] =!= CA] :=
    Module[{z}, z = x; Part[z, Sequence @@ Table[1, {p + q}]] += y;
    CA[p, q, z]] /; p + q == carank[x];
```

```
(* The product formula uses the recursive algorithm implemented above. *)

CA /: CA[p_Integer /; p >= 0, q_Integer /; q >= 0, x_ /; catest[x]] **
    CA[p2_Integer /; p2 >= 0, q2_Integer /; q2 >= 0, y_ /; catest[y]] :=
    CA[p, q, prod[p, q, x, y]] /; And[p + q == carank[x] == carank[y],
    p == p2, q == q2];

(* The following three definitions allow multiplication with ordinary numbers. *)

CA /: CA[p_Integer /; p >= 0, q_Integer /; q >= 0, x_ /; catest[x]] *
    y_ /; And[Head[y] =!= List, Head[y] =!= CA] :=
    CA[p, q, x y] /; p + q == carank[x];

CA /: CA[p_Integer /; p >= 0, q_Integer /; q >= 0, x_ /; catest[x]] **
    y_ /; And[Head[y] =!= List, Head[y] =!= CA] :=
    CA[p, q, x y] /; p + q == carank[x];

CA /: y_ ** CA[p_Integer /; p >= 0, q_Integer /; q >= 0, x_ /; catest[x]] /;
    And[Head[y] =!= List, Head[y] =!= CA] := CA[p, q, x y] /; p + q == carank[x];

(* We also give a definition for the Clifford product if both factors are numbers. *)

Unprotect[NonCommutativeMultiply];

NonCommutativeMultiply[x_ /; And[Head[x] =!= List, Head[x] =!= CA,
    !MatchQ[x, Subscript[\[CapitalGamma], _]]],
    y_ /; And[Head[y] =!= List, Head[y] =!= CA,
    !MatchQ[y, Subscript[\[CapitalGamma], _]]]] := x y;

NonCommutativeMultiply[x_] := x;

Protect[NonCommutativeMultiply];

(* Involution. *)

CAAlpha[CA[p_Integer /; p >= 0, q_Integer /; q >= 0, x_ /; catest[x]]] :=
    CA[p, q, alpha[x]] /; p + q == carank[x];

(* Transpose. *)

CATranspose[CA[p_Integer /; p >= 0, q_Integer /; q >= 0, x_ /; catest[x]]] :=
    CA[p, q, transpose[x]] /; p + q == carank[x];

(* Conjugate. *)

CAConjugate[CA[p_Integer /; p >= 0, q_Integer /; q >= 0, x_ /; catest[x]]] := CA[p, q,
    If[p + q == 0, Conjugate[x], Array[Power[-1, Plus @@ Take[List[##] - 1, p]]&,
    Table[2, {p + q}]] Conjugate[x]]] /; p + q == carank[x];
```

```
(* Adjoint.*)

CAAdjoint[CA[p_Integer /; p >= 0, q_Integer /; q >= 0, x_ /; catest[x]]] :=
    CAConjugate[CATranspose[x]] /; p + q == carank[x];

(* The inverse of an ordinary number. *)

inverse[0, 0, x_ /; catest[x]] := 1 / x /; carank[x] == 0;

(* Inversion formula, part 1. *)

inverse[0, q_Integer /; q > 0, x_ /; catest[x]] := Module[{a, b, c, d, y},
    a = x[[1]]; b = x[[2]]; If[And @@ ((# === 0)& /@ Flatten[{a}]),
    y = inverse[0, q - 1, b];
    d = inverse[0, q - 1, -prod[0, q - 1, a, prod[0, q - 1, y, alpha[a]]] + alpha[b]];
    c = -prod[0, q - 1, y, prod[0, q - 1, alpha[a], d]],
    y = inverse[0, q - 1, alpha[a]];
    c = inverse[0, q - 1, a - prod[0, q - 1, alpha[b], prod[0, q - 1, y, b]]];
    d = -prod[0, q - 1, y, prod[0, q - 1, b, c]]];
    Simplify[{c, d}]] /; q == carank[x];

(* Inversion formula, part 2. *)

inverse[p_Integer /; p > 0, q_Integer /; q >= 0, x_ /; catest[x]] :=
    Module[{a, b, c, d, y},
    a = x[[1]]; b = x[[2]]; If[And @@ ((# === 0)& /@ Flatten[{a}]),
    y = inverse[p - 1, q, b];
    d = inverse[p - 1, q, -prod[p - 1, q, a, prod[p - 1, q, y, alpha[a]]] - alpha[b]];
    c = -prod[p - 1, q, y, prod[p - 1, q, alpha[a], d]],
    y = inverse[p - 1, q, alpha[a]];
    c = inverse[p - 1, q, a + prod[p - 1, q, alpha[b], prod[p - 1, q, y, b]]];
    d = -prod[p - 1, q, y, prod[p - 1, q, b, c]]];
    Simplify[{c, d}]] /; p + q == carank[x];

(* To make the inverse more useful, wrap it. *)

CAInverse[CA[p_Integer /; p >= 0, q_Integer /; q >= 0, x_ /; catest[x]]] :=
    CA[p, q, inverse[p, q, x]] /; p + q == carank[x];

(* Powers are computed recursively. *)

CA /: Power[CA[p_Integer /; p >= 0, q_Integer /; q >= 0, x_ /; catest[x]],
    n_Integer /; n > 1] := Power[CA[p, q, x], n - 1] ** CA[p, q, x] /;
    p + q == carank[x];

CA /: Power[CA[p_Integer /; p >= 0, q_Integer /; q >= 0, x_ /; catest[x]], 1] :=
    CA[p, q, x] /; p + q == carank[x];
```

```
CA /: Power[CA[p_Integer /; p >= 0, q_Integer /; q >= 0, x_ /; catest[x]], 0] :=
    CA[p, q, one[p + q]] /; p + q == carank[x];

CA /: Power[CA[p_Integer /; p >= 0, q_Integer /; q >= 0, x_ /; catest[x]],
    n_Integer /; n < 0] :=  Power[CA[p, q, inverse[p, q, x]], -n];

(* Right division: x / y = xy^-1 *)

CA /: Divide[CA[p_Integer /; p >= 0, q_Integer /; q >= 0, x_ /; catest[x]],
    CA[p2_Integer /; p2 >= 0, q2_Integer /; q2 >= 0, y_ /; catest[y]]] :=
    CA[p, q, x] ** inverse[CA[p, q, y]] /;
    And[p + q == carank[x] == carank[y], p == p2, q == q2];

(* These two formulas allow right division with ordinary numbers. *)

CA /: Divide[CA[p_Integer /; p >= 0, q_Integer /; q >= 0, x_ /; catest[x]],
    y_ /; And[Head[y] =!= List, Head[y] =!= CA]] := CA[p, q, x / y] /;
    p + q == carank[x];

CA /: Divide[x_ /; And[Head[x] =!= List, Head[x] =!= CA],
    CA[p_Integer /; p >= 0, q_Integer /; q >= 0, y_ /; catest[y]]] :=
    x inverse[CA[p, q, y]] /; p + q == carank[y];

(* Left division: x \ y = x^-1 y *)

CA /: Backslash[CA[p_Integer /; p >= 0, q_Integer /; q >= 0, x_ /; catest[x]],
    CA[p2_Integer /; p2 >= 0, q2_Integer /; q2 >= 0, y_ /; catest[y]]] :=
    inverse[CA[p, q, x]] ** CA[p, q, y] /;
    And[p + q == carank[x] == carank[y], p == p2, q == q2];

(* These two formulas allow left division with ordinary numbers. *)

CA /: Backslash[CA[p_Integer /; p >= 0, q_Integer /; q >= 0, x_ /; catest[x]],
    y_ /; And[Head[y] =!= List, Head[y] =!= CA]] := inverse[CA[p, q, x]] y /;
    p + q == carank[x];

CA /: Backslash[x_ /; And[Head[x] =!= List, Head[x] =!= CA],
    CA[p_Integer /; p >= 0, q_Integer /; q >= 0, y_ /; catest[y]]] :=
    CA[p, q, y / x] /; p + q == carank[y];

(* Finally, all Clifford are printed in the standard basis. *)

Format[CA[p_Integer /; p >= 0, q_Integer /; q >= 0, x_ /; catest[x]]] :=
    Module[{pl, i, j, t},   pl = Position[x, _, {p + q}, Heads -> False];
    t = Table[1, {Length[pl]}]; Do[Do[If[pl[[i,j]] == 2,
    If[t[[i]] === 1, t[[i]] = Subscript[\[CapitalGamma], ToString[j]],
    t[[i,2]] = StringJoin[t[[i,2]], ",", ToString[j]]]], {j, 1, p + q}];
    t[[i]] = Part[x, Sequence @@ pl[[i]]] t[[i]], {i, 1, Length[pl]}]; Plus @@ t] /;
    carank[x] == p + q;
```

```
End[];
EndPackage[];
```

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Ich versichere, diese Arbeit selbstständig und nur unter Benutzung der angegebenen Hilfsmittel und Quellen angefertigt zu haben. Ich gestatte die Veröffentlichung dieser Arbeit.

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