The role of symmetry in the geometric description of gravity

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Abstract

To our current level of understanding, gravity is most appropriately described in terms of differential geometry. This understanding has evolved more than a century ago with the advent of general relativity. Since then it has been an important paradigm in the development of further gravity theories, which can be seen as modifications or extensions of general relativity, and whose aim is to address the remaining open questions in our knowledge on gravity. These arise in particular from the tensions of general relativity with observations in cosmology, which necessitate the introduction of additional, "dark", and so far unexplained components of the universe, as well as its tensions with quantum theory. These open questions have stipulated the development of numerous gravity theories, whose underlying geometric description differs from general relativity, and much effort has been invested to study both the geometric foundations and phenomenology of these theories.

Whenever geometry is employed in the description of a physical theory, symmetry attains an important role. Prominent examples beyond the realm of gravity theory are Galilei and Lorentz invariance in classical mechanics and special relativity, gauge symmetries in the standard model of particle physics and crystallographic symmetries in solid state physics. This is also the case in gravity theory. One of the most common strategies in order to simplify the field equations of a gravity theory under consideration by reducing their independent number of components is to restrict one's attention to solutions which are invariant under the action of a transformation group. While this procedure is straightforward for gravity theories which use well-known geometric notions in their description, such as the metric tensor in general relativity, it is yet to be explored for gravity theories which make use of more general geometric frameworks. Going beyond these exactly symmetric solutions, it is common to study perturbations around such solutions. In this case the symmetries of the background leave an imprint on the structure of the perturbations, and lead again to a simplification. Finally, besides studying symmetric solutions of gravity theories, one may also study the symmetries of the theories themselves, very much akin to the study of gauge symmetries in particle physics.

The aim of this thesis is to bring together nine articles which are devoted to the study and application of the aforementioned incantations of symmetries in the geometric description of gravity theories. Two of these articles are devoted to the extension of the notion of symmetry, by which we understand invariance of a particular geometry under the action of a transformation group, to Cartan and teleparallel geometries, both of which are employed in gravity theory. The next two articles make use of these notions of symmetry in order to derive the most general solutions with spherical and cosmological symmetry for different geometries. Beyond these exact solutions, three more articles discuss perturbations around symmetric backgrounds and the transformation of more general solutions under the action of the symmetry group. Finally, the last two articles discuss the transformation of gravity theories under the action of a transformation group on their field space.

Zusammenfassung

Unserem derzeitigen Kenntnisstand nach wird die Gravitation am zutreffendsten durch Differentialgeometrie beschrieben. Diese Erkenntnis ist vor über einem Jahrhundert mit der Entwicklung der allgemeinen Relativitätstheorie zustande gekommen. Seitdem ist sie zu einem wichtigen Paradigma für die Entwicklung weiterer Gravitationstheorien geworden, die als Veränderungen oder Erweiterungen der allgemeinen Relativitätstheorie betrachtet werden können, und deren Ziel es ist, die noch offenen Fragen in unserem Wissen über die Gravitation anzugehen. Diese beruhen insbesondere auf den Spannungen zwischen der allgemeinen Relativitätstheorie und Beobachtungen in der Kosmologie, die die Einführung von weiteren, "dunklen", und bisher unerklärten Komponenten des Universums erfordern, sowie den Spannungen gegenüber der Quantentheorie. Diese offenen Fragen haben zur Entwicklung zahlreicher Gravitationstheorien geführt, deren zugrundeliegende geometrische Beschreibung sich von der der allgemeinen Relativitätstheorie unterscheidet, und intensive Bemühungen wurden unternommen um sowohl die geometrischen Grundlagen als auch die Phänomenologie dieser Theorien zu untersuchen.

Wenn Geometrie für die Beschreibung einer physikalischen Theorie herangezogen wird, kommt der Symmetrie eine besondere Rolle zu. Bekannte Beispiele außerhalb des Fachgebiets der Gravitationstheorie sind die Galilei- und Lorentz-Invarianz in der klassischen Mechanik und der speziellen Relativitätstheorie, Eichsymmetrien im Standardmodell der Teilchenphysik und kristallographische Symmetrien in der Festkörperphysik. Dies ist auch in der Gravitationstheorie der Fall. Eine der am häufigsten angewandten Strategien um die Feldgleichungen der betrachteten Gravitationstheorie durch eine Verringerung ihrer unabhängigen Komponenten zu vereinfachen besteht darin, sich auf Lösungen zu beschränken, die invariant unter der Wirkung einer Transformationsgruppe sind. Während diese Prozedur unmittelbar auf Gravitationstheorien anwendbar ist, deren Beschreibung auf bekannte geometrische Begriffe zurückgreift, wie den metrischen Tensor in der allgemeinen Relativitätstheorie, muss sie noch untersucht werden für Theorien, denen ein allgemeinerer geometrischer Rahmen zugrunde liegt. Über diese exakten Lösungen hinaus werden auch Störungen um solche Lösungen herum untersucht. In diesem Fall haben die Symmetrien der Hintergrundlösung auch Auswirkungen auf die Struktur der Störungen, und führen so wiederum zu einer Vereinfachung. Zum Abschluss ist es auch möglich, außer symmetrischer Lösungen von Gravitationstheorien, auch die Symmetrien der Theorien an sich zu untersuchen, in ähnlicher Weise wie bei den Eichtheorien in der Teilchenphysik.

Ziel dieser Arbeit ist die Zusammenfassung von neun Fachartikeln, die sich der Untersuchung und Anwendung der genannten Ausführungen von Symmetrien in der geometrischen Beschreibung von Gravitationstheorien widmen. Zwei dieser Artikel widmen sich der Erweiterung des Symmetriebegriffs, worunter wir die Invarianz einer gegebenen Geometrie unter der Wirkung einer Transformationsgruppe verstehen, auf Cartan- und teleparallele Geometrie, die beide in der Gravitationstheorie zur Anwendung kommen. Die nächsten beiden Artikel machen von diesem Symmetriebegriff Gebrauch, um die allgemeinsten Lösungen mit sphärischer und kosmologischer Symmetrie für verschiedene Geometrien zu bestimmen. Jenseits dieser exakten Lösungen befassen sich drei weitere Artikel mit Störungen um diese exakten Lösungen und die Transformation allgemeinerer Lösungen unter der Wirkung der Symmetriegruppe. Die verbleibenden beiden Artikel befassen sich mit der Transformation von Gravitationstheorien unter der Wirkung einer Transformationsgruppe auf ihrem Feldraum.

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Contents

0 Scientific work by the author

The scientific work of the author focuses on the geometric foundations of various modified theories of gravity, as well as their comparison to observations. Naturally, it can be characterized either by the theories under consideration, or by the aspects which have been studied. We do the former in section 0.1 and the latter in section 0.2.

0.1 Gravity theories

The following classes of gravity theories have been studied by the author of this thesis:

Multi-metric theories: The main part of the author's PhD thesis [56] was dedicated to the construction of a multi-metric theory of gravity which mediates a repulsive gravitational force between different types of masses in the Newtonian limit, in order to address the open problem of the accelerating expansion of the universe. This has shown to be impossible with $N = 2$ metrics [100], but yields an accelerating cosmology [102] with $N > 2$ while being consistent with solar system [101] and gravitational wave [57] observations, as well as structure formation [55], and allows for the existence of wormholes [61]. The multi-metric post-Newtonian formalism developed for studying the field limit has later been extended [60] and applied to ghost-free bimetric gravity [68].

Finsler and Cartan geometry: Most gravity theories are manifestly Lorentz invariant by employing a pseudo-Riemannian spacetime metric as their fundamental field. A consequence of this assumption is a restrictive, Lorentz invariant form of the dependence of the gravitational interaction on the velocities of source and test masses. Approaches to quantum gravity, however, suggest an alteration of this behavior at very high energies. A more general velocity dependence can be achieved by lifting the gravitational interaction from the spacetime manifold itself to certain fiber bundles, which encode the position, velocity and possibly even the spatial frame components of an observer. Gravity theories based on this approach can be implemented using Finsler or Cartan geometry. The latter can be used as a fundamental geometry for gravity theories [118], and has further been shown to encompass certain types of Finsler spacetime geometries [58, 59], which can be exploited in order to derive a unified notion of spacetime symmetries [65, 66]. Finsler geometry provides interesting possibilities as it provides the natural background geometry for modeling fluid dynamics [63, 67] through the formalism of kinetic gases. More recently, it has been shown that the same formalism can be used to model kinetic gases as sources of gravity [95, 96]. Together with a consistent Finsler gravity theory [93, 98] this allows to study Finsler geometry as a novel model, e.g., for cosmology [88, 94]. To study Finsler geometry with spherical symmetry, a new type of harmonics has been proposed [73].

Teleparallel gravity theories: While general relativity describes gravity as the curvature of the symmetric, metric-compatible Levi-Civita connection on spacetime, one may also consider the opposite assumption and employ a flat, curvature free connection, which is not symmetric or not metric-compatible, or both. Theories of these types are called teleparallel gravity theories. In the case that the connection is still metric-compatible, it is fully characterized by its torsion. Various phenomenological aspects of this class of theories have been studied, such as its cosmology [83], gravitational waves [87] or post-Newtonian limit [158]. Further, also the theoretical consistency of teleparallel gravity theories has been studied by making use of the Hamilton formalism [17, 18, 16, 75, 15]. Other foundational questions which have been studied involve the structure of the gravity action compared to electrodynamics [84], the definition of symmetry [85] and its application to exact solutions [110, 77] and perturbation theory [78]. Similarly, for theories with a symmetric teleparallel connection, in which only nonmetricity is present, gravitational waves [91], the post-Newtonian limit [39] and the analogy with electrodynamics [117] have been studied. Finally, also for theories in which several of these geometries are present, gravitational waves [69], spherical symmetry [76] and variational calculus [79] have been discussed. In addition to this original research, the thesis author has contributed to a topical review [6] and a chapter on teleparallel gravity in another review [150].

Scalar and multi-scalar extensions: A common approach to modify a gravity theory is by adding a single or multiple scalar fields, together with additional couplings to the fields which mediate the gravitational interaction. Besides being postulated as fundamental theories, such scalar extensions also arise as effective models from modified action functionals, higher dimensional theories or approaches to quantum gravity. Scalar extensions of general relativity are known as scalar-tensor, or more precisely scalar-curvature theories of gravity. The work of the thesis author encompasses in particular the post-Newtonian limit of theories with one scalar field [81, 99] or multiple scalar fields [82], as well as including higher order coupling terms in the Horndeski class of theories [62, 64]. For theories which can be represented as scalar-curvature theories, wormholes [92] and the variational principle [97] have been studied. Besides curvature-based gravity theories, also teleparallel gravity theories can be extended with scalar fields. The covariant formulation of such scalar-torsion theories was put forward in [86], and studied in full detail in a series of articles [70, 89, 71]. While these theories allow for conformal transformations, including additional derivative couplings allows to generalize this class to disformal transformations [72]. Also for such scalar-torsion theories of gravity the post-Newtonian limit has been calculated [32, 38]. This study has been generalized to a teleparallel generalization of Horndeski gravity [7], for which also gravitational waves were studied [5]. Further, a new class of pseudoscalar fields in teleparallel gravity was proposed and studied in cosmology [90].

0.2 Studied topics

The scientific work of the thesis author may also be categorized by the different aspects of gravity theories which have been studied. The following topics have been studied most intensively:

Cosmology: Since cosmology is one of the main motivations for considering modified gravity theories, it has been the topic of various articles by the thesis authors. The cosmological dynamics of a homogeneous and isotropic universe have been studied in the context of multi-metric theories [102], Finsler geometry [63, 67, 88, 94, 96] and teleparallel gravity [83, 86, 77, 78, 90]; for the latter theory it has also been a major part of a topical review [6].

Post-Newtonian formalism: The post-Newtonian formalism has been applied to a number of gravity theories, and extended to encompass the more general underlying geometry employed by these theories. This concerns in particular theories with multiple metrics [101, 60, 68], scalar-curvature theories [81, 62, 64, 82, 99] and various flavors of teleparallel theories [158, 32, 38, 7, 39]. Further, the formalism itself has been studied in order to simplify its application, by devising a gauge-invariant approach [74] and developing a dedicated tensor algebra package [80]. The parametrized post-Newtonian formalism and its extension has also been the topic of a chapter contributed to a major review article [150].

Gravitational waves: Since their first observation, gravitational waves have become an important test case for modified gravity theories. Their propagation has been studied in multi-metric gravity theories [57] and various types of teleparallel gravity theories [69, 87, 91, 5].

Symmetry transformations: One of the most important research topics in the work of the thesis author, which is also most relevant for this thesis, is that of symmetry transformations, their derivation and application in gravity theories. In particular, notions of spacetime symmetries have been derived using Cartan geometry [65, 66]. They have been used to derived particularly symmetric geometries with spherical symmetry [76] and cosmological symmetry [77]. Further, they have been used to develop gauge-invariant formalisms for perturbations [74, 78] and to construct spherical harmonics in Finsler geometry [73]. Finally, symmetries of the field space of scalar-torsion theories have been used to construct families of actions which are invariant under conformal transformations [70, 89, 71] or disformal transformations [72].

Hamilton formalism: An important tool to study the consistency of physical theories is the Hamilton formalism, which allows to deduce and number and nature of degrees of freedom of a theory under consideration. This formalism has been used to study teleparallel gravity theories [17, 18, 16, 75, 15]

Other exact solutions: Among the most peculiar solutions found in modified gravity theories are wormholes; such solutions have been found in multi-metric gravity [61] and conformal gravity [92]. Another class of considered solutions are rotating spacetimes in teleparallel gravity [110].

Other foundational issues: Different principles are central in the construction of physical field theories. Most commonly, theories are derived from an action principle; however, one often finds that different actions and choices of dynamical variables may yield the same field equations, and thus an equivalent theory. This has been shown for various teleparallel gravity theories [79]. The opposite question is posed by the inverse problem of variational calculus: given a set of field equations, are they variational, and how can one find an action? This problem has been investigated for Finsler gravity [93, 98] and 4-dimensional Gauss-Bonnet gravity [97]. Another possible starting point for the construction of a gravity theory is similar to the axiomatic approach to electrodynamics, and the central ingredient is a constitutive relation. This has been studied for different types of teleparallel gravity theories [84, 117].

1 Introduction

The development of the general theory of relativity [31] is often celebrated as laying the foundations for the geometric description of gravity. While in Newton's theory [134] gravity is modeled as a force which acts instantaneously between distant bodies, and space and time are merely the stage on which these bodies are located, in general relativity spacetime itself becomes an important actor, mediating the gravitational interaction through its pseudo-Riemannian geometry. This geometry defines spatial distances and the proper time measured by clocks through the length functional of the metric, and the motion of test masses through the autoparallels of the associated Levi-Civita connection, hence closely linking these different physical quantities. This geometric description of gravity within the general theory of relativity has excelled in its ability to model numerous gravitating systems, including the solar system [160] and gravitational waves [14].

Despite its success, general relativity is challenged by precision observations in cosmology [12] as well as its tensions with quantum physics [147, 148], which have so far obstructed the development of a full, consistent quantum gravity theory, and the unification of all four fundamental forces. These open questions have stipulated the development of a plethora of modified gravity theories; see [151] for a recent review. The majority of these theories follows the same spirit as general relativity, modeling the gravitational interaction via the geometry of spacetime. The geometric objects used to describe this geometry, however, usually differ from the pseudo-Riemannian geometry employed by general relativity. The necessity to modify the mathematical foundation of the theory in order to construct modifications of general relativity is rooted in Lovelock's theorem, which implies that general relativity is the unique local, Lagrangian field theory in four dimensions, whose only dynamical variable is the metric tensor, and whose field equations are of second derivative order [129, 130]. Retaining the variational principle, locality, a four-dimensional spacetime and second-order field equations to avoid ghost instabilities [138], one is therefore forced to consider alternative geometries. Common modifications include adding scalar fields [43], vector fields [54], additional metrics [52] or different affine connections [53, 109, 10, 22], or to generalize the metric length functional [142, 140].

Given a theory of gravity, one of the most natural arising questions is for the solutions to its field equations. Among the first solutions of Einstein's equations of general relativity are the vacuum solution of the exterior of a spherically symmetric gravitating body [153, 29] and of a homogeneous, isotropic universe [41, 42, 127, 128, 126]. Both have in common that they exhibit a high amount of symmetry, where the latter is understood as the action of a transformation group on spacetime which leaves the matter distribution and geometry invariant. Infinitesimally, this action can be described by a number of vector fields, which act on the dynamical fields of the theory by the Lie derivative. For the pseudo-Riemannian geometry used in general relativity, the vector fields which generate symmetries of the geometry are known as Killing vector fields. For other geometries, more general notions of symmetry and invariance under group actions appear [166].

Besides being instrumental in finding solutions to the field equations, the notion of symmetry has an invaluable role in the geometric description of physical field theories. One of the most important manifestations of this role is given by Noether's theorems, which relate the symmetries of a Lagrangian theory to the existence of conserved quantities and constraint equations [135]. Looking back to the aforementioned examples of spherically and cosmologically symmetric gravitational fields, the symmetry present in these geometries leads to the conservation of angular momentum, and in the cosmological case also linear momentum, of test masses. Here the Lagrangian to which Noether's theorem is applied is that of the test mass, while the spacetime geometry appears as the background which determines its motion. However, if also gravity itself is modeled by a Lagrangian fields theory, Noether's theorem can applied as well, where in this case the transformation group generating the symmetry does not act on spacetime, but on the space of values for the fields which describe the considered gravity theory. This principle gives rise to the notion of gauge symmetries. Moreover, considering not a single field theory, but a family of theories defined by a set of parameters, the action of a transformation group may be used to relate different constituents of this family, hence showing their physical equivalence, up to a change of variables.

The following topics are presented in more detail. In section 2, we briefly review the most important ingredients for the geometric description of gravity and the notion of symmetry. We show how this notion gives conditions for symmetries in different types of geometries, which are relevant in the context of gravity theory, in section 3. In section 4, we apply these conditions to several classes of geometry, and show how symmetric gravitational backgrounds arise as solutions of these conditions. These exact background geometries serve as the foundation for perturbations in section 5, where we focus on particular on the action of transformation groups on these perturbations. While the aforementioned sections concern the action of symmetry groups on the spacetime manifold, we also consider symmetries of the field space of such theories, and discuss the resulting transformations in section 6. We give a brief summary and outlook in section 7.

2 Preliminaries

Before coming to the scientific results presented and discussed in this thesis, we provide a brief review of the most relevant notions we make use of. For a detailed exposition, The main aim of this section is to set the conventions for terminology and notation which we will use throughout this thesis; for a more thorough introduction to these topics, the reader is referred to the respective literature. We start by providing the definitions of the most relevant notions from differential geometry in section 2.1. These are used to describe the most intensively studied classes of geometries in section 2.2. Finally, in section 2.3 we give a general overview of the notions of symmetry we employ in this thesis.

2.1 Notions from differential geometry

In the geometric description of gravity, the notion of geometry refers to *differential* geometry in the vast majority of cases. Although we assume that the reader is familiar with this field of mathematics, we summarize here the most important notions we employ, their definitions in the conventions we use and our notation. In particular, we discuss fiber bundles and bundle morphisms in section 2.1.1, natural bundles in section 2.1.2, jet bundles in section 2.1.3, connection bundles in section 2.1.4 and, most importantly for the study of symmetries, diffeomorphisms and the Lie derivative in section 2.1.5. For a general introduction, see e.g. [116].

2.1.1 Fiber bundles and bundle morphisms

The most important notion in field theory and the mathematical description of gravity theories is that of a *fiber bundle*, which is defined as follows.

Definition 2.1 (Fiber bundle)**.** A *fiber bundle* (E, B, π, F) consists of manifolds E, B, F and a surjective map $\pi : E \to B$, such that for any $b \in B$ there exists an open set $U \subset B$

containing b and a diffeomorphism $\phi : \pi^{-1}(U) \to U \times F$ such that the diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow[\phi]{} & U \times F \\ \downarrow{\scriptstyle \pi} & \swarrow{\scriptstyle \mathrm{pr}_1} & \\ U & & \end{array} \tag{2.1.1}$$

commutes.

The manifolds in the construction above are called the *base* B, the *total space* E and *typical fiber* F, while π is called the *projection* or *bundle map*. The preimage $\pi^{-1}(b)$ of b is called the *fiber* over b, and we will also denote it by E_b. Further, we call a *section* a map $\sigma : B \to E$ such that $\pi \circ \sigma = \mathrm{id}_B$ is the identity on B. The latter condition means that $\sigma(b) \in E_b$ for all $b \in B$. The space of all sections will be denoted by $\Gamma(\pi)$. Sometimes it will be more convenient to consider *local sections* $\sigma : U \to E$ instead, whose domain $U \subset B$ is an open subset of B, and which satisfy $\pi \circ \sigma = \mathrm{id}_U$.

Different fiber bundles are related by the following notion.

Definition 2.2 (Bundle morphism)**.** Let $\pi_1 : E_1 \to B_1$ and $\pi_2 : E_2 \to B_2$ be fiber bundles. A *bundle morphism* is a map $\Psi : E_1 \to E_2$ such that there exists a map $\psi : B_1 \to B_2$ for which the diagram

$$\begin{array}{ccc} E_1 & \xrightarrow{\Psi} & E_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ B_1 & \xrightarrow[\psi]{} & B_2 \end{array} \tag{2.1.2}$$

commutes. The bundle morphism Ψ is then said to *cover* ψ.

Note that it is not necessary to specify the map ψ between the base manifolds explicitly, as it is already uniquely determined by the map Ψ. To see this, recall that by definition of a fiber bundle, the projection π_1 must be surjective. Hence, for $b \in B_1$, there exists $e \in E_1$, such that $b = \pi_1(e)$. One can then use the commutativity of the diagram (2.1.2) to derive

$$\psi(b) = \psi(\pi_1(e)) = \pi_2(\Psi(e)) \,, \tag{2.1.3}$$

where the right hand side is determined by Ψ. From this relation further follows that if $e, e' \in E_1$ lie in the same fiber, so that $\pi_1(e) = \pi_2(e)$, one has

$$\pi_2(\Psi(e)) = \psi(\pi_1(e)) = \psi(\pi_1(e')) = \pi_2(\Psi(e')) \,. \tag{2.1.4}$$

Hence, a bundle morphism preserves the fibers.

In this work, we are mostly interested in bundle morphisms, whose domain and codomain are the same fiber bundle $\pi : E \to B$. Such bundle morphisms are also denoted *bundle automorphisms*. In particular, a bundle automorphism which covers the identity id_B on the base manifold is called *vertical*.

2.1.2 Natural bundles

In the geometric description of gravity theories, one usually makes use of a particular class of fiber bundles, called *natural bundles*, which "naturally" arise from their base manifolds. More formally, this notion can be defined by making use of category theory [30, 131]. In this context, it is helpful to note that smooth manifolds and smooth maps between them form a category $\mathbf{Man}^\infty$. For the definition of natural bundles, it turns out to be useful to make the following identifications:

1. The *objects* of the category are smooth manifolds M.

2. The *morphisms* of the category are smooth embeddings $\psi : M \to N$ between manifolds M, N of the same dimension $\dim M = \dim N$.

3. The *source* of a map $\psi : M \to N$ is its domain M.

4. The *target* of a map $\psi : M \to N$ is its codomain N.

5. The *composition* of morphisms is map composition, $\varphi \circ \psi$ for $\psi : M \to N$ and $\varphi : N \to O$.

Note that we have restricted ourselves to very specific make between manifolds, which is important for the construction below. Similarly, also fiber bundles form a category $\mathbf{Fib}^\infty$, whose objects are fiber bundles, morphisms are fiber bundle morphisms (where we make the same restriction as for the category $\mathbf{Man}^\infty$), and the source, target and composition of morphisms are the domain, codomain and map composition. Finally, we mention that there exists a functor $\mathscr{B} : \mathbf{Fib}^\infty \to \mathbf{Man}^\infty$, called the *base functor*, which assigns to every fiber bundle its base manifold, and to every fiber bundle morphism the underlying map on the base manifold; similarly, a total space functor $\mathscr{E} : \mathbf{Fib}^\infty \to \mathbf{Man}^\infty$ exists.

With these notions in place, one may define a natural bundle as a *bundle functor*, which is defined as a functor $\mathscr{F} : \mathbf{Man}^\infty \to \mathbf{Fib}^\infty$ satisfying a number of properties [139, 119]:

1. For every manifold M, $\mathscr{F}(M)$ is a fiber bundle over M.

2. For every smooth embedding $\psi : M \to N$ between manifolds M, N of the same dimension $\dim M = \dim N$, $\mathscr{F}(\psi) : \mathscr{F}(M) \to \mathscr{F}(N)$ is a bundle morphism covering ψ.

These two conditions can also be expressed by making use of the base functor $\mathscr{B} : \mathbf{Man}^\infty \to \mathbf{Fib}^\infty$ we mentioned above. A bundle functor $\mathscr{F}$ must satisfy $\mathscr{B} \circ \mathscr{F} = \mathscr{I}_{\mathbf{Man}^\infty}$, where $\mathscr{I}_{\mathbf{Man}^\infty}$ is the identity functor on $\mathbf{Man}^\infty$. We will give a few examples below, where it will be understood that $\psi : M \to N$ is a map satisfying certain conditions, even through we will not repeat them for brevity.

The notions above can be illustrated with an intuitive example, given by the *tangent functor* $\mathscr{T} : \mathbf{Man}^\infty \to \mathbf{Fib}^\infty$, which is defined as follows:

1. To every manifold M, $\mathscr{T}$ assigns the tangent bundle $\tau_M : TM \to M$.

2. To every smooth map $\psi : M \to N$, $\mathscr{T}$ assigns the bundle morphism $\psi_* = \mathrm{D}\psi : TM \to TN$, known as the differential of ψ, given by the Jacobian of ψ at every point of M.

The tangent bundle is a special case of the more general notion of tensor bundles of rank (r, s), where $r, s \in \mathbb{N}$. In a similar fashion, they are natural bundles which can be described by a functor $\mathscr{T}_s^r$, such that:

1. To every manifold M, $\mathscr{T}_s^r$ assigns the tensor bundle $\tau^r_{s\,M} : T_s^r M \to M$.

2. To every smooth map $\psi : M \to N$, $\mathscr{T}_s^r$ assigns the bundle morphism $\bigotimes^r \mathrm{D}\psi \otimes \bigotimes^s \mathrm{D}\psi^{-1} : T_s^r M \to T_s^r N$, where $\mathrm{D}\psi^{-1}$ is the inverse of the Jacobian at every point in M.

Note the appearance of the inverse $\mathrm{D}\psi^{-1}$ of the Jacobian, which is the reason for restricting the morphisms ϕ of the category $\mathbf{Man}^{\infty}$. Further note that a special case is given by $r = s = 0$; the corresponding tensor bundle is the trivial bundle $M \times \mathbb{R}$. This can also be viewed as a special case of another construction, which assigns to every manifold M is product with a fixed manifold F, i.e., the trivial bundle $M \times F$ with fiber F. For given fiber space F, one can thus define the trivial bundle functor $\mathscr{P}_F$ as follows:

1. To every manifold M, $\mathscr{P}_F$ assigns the trivial bundle $\mathrm{pr}_1 : M \times F \to M$, where pr_1 denotes the projection onto the first factor.

2. To every smooth map $\psi : M \to N$, $\mathscr{P}_F$ assigns the bundle morphism $(\psi, \mathrm{id}_F) : M \times F \to N \times F, (p, q) \mapsto (\psi(p), q)$.

Hence, also trivial fiber bundles can be regarded as natural bundles.

2.1.3 Jet bundles

A class of fiber bundles which is of particular use in the description of physical theories are jet bundles. They provide the mathematical foundation for the commonly used concept that a function depends on a physical quantity and its derivatives up to a given, finite order, which is used, e.g., in Lagrange theory. They are defined as follows [152].

Definition 2.3 (Jet bundle)**.** Let $\pi : E \to B$ be a fiber bundle, $U \subset B$ and $\sigma : U \to E$ a local section, $\pi \circ \sigma = \mathrm{id}_U$. For $r \in \mathbb{N}$ and a point $b \in U$, we define the *r-jet* $j^r_b\sigma$ of σ at b as the equivalence class of local sections $\tau : V \to E$, where $b \in V \subset B$, where σ and τ are regarded equivalent if and only if for all smooth curves $\gamma : \mathbb{R} \to U \cap V$ with $\gamma(0) = b$ and smooth functions $f : E \to \mathbb{R}$ the functions $f \circ \sigma \circ \gamma$ and $f \circ \tau \circ \gamma$ agree in their derivatives up to order r at 0, i.e.,

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k}(f \circ \sigma \circ \gamma)(t)\bigg|_{t=0} = \frac{\mathrm{d}^k}{\mathrm{d}t^k}(f \circ \tau \circ \gamma)(t)\bigg|_{t=0} \tag{2.1.5}$$

for all $0 \leq k \leq r$. The space of all r-jets at $b \in B$ is denoted $J^r_b(\pi)$, while the space of all r-jets is denoted $J^r(\pi)$.

Given local coordinates (x^μ, y^a) on E, where (x^μ) are local coordinates on B and (y^a) are fiber coordinates, such that the bundle map is given by $\pi : (x^\mu, y^a) \mapsto (x^\mu)$, one may in particular choose γ to be a coordinate line on B, and f a fiber coordinate on E. It then follows from its definition that two sections which define the same jet must have identical Taylor coefficients up to order r. This condition is not only necessary, but also sufficient, since also any other curve γ and function f can be expressed using these local coordinates. Hence, the coordinate expression of a jet $j^r_b\sigma$ may be identified with the Taylor coefficients of the coordinate expression of σ up to order r at the point $b \in B$. These Taylor coefficients may therefore be used as fiber coordinates on the jet bundle $J^r(\pi)$, and so give rise to local induced coordinates

$$(x^\mu, y^a, y^a{}_{\mu_1}, \ldots, y^a{}_{\mu_1\cdots\mu_r}), \tag{2.1.6}$$

where $\mu_1 \leq \ldots \leq \mu_r$, on the total space $J^r(\pi)$. They are also instrumental in the coordinate descriptions of the projections $\pi_{r,s} : J^r(\pi) \to J^s(\pi)$ with $r > s$, which reduces the order of the Taylor expansion, with the special case $J^0(\pi) = E$, and $\pi_r = \pi \circ \pi_{r,0} : J^r(\pi) \to B$. Finally, it is helpful to note that if $\pi : E \to B$ is a natural bundle over B, defined by a bundle functor $\mathscr{F}$, then also the jet bundles $\pi_r : J^r(\pi) \to B$ are natural bundles, i.e., they are defined by a jet bundle functor $\mathscr{J}^r_{\mathscr{F}}$.

2.1.4 Connection bundles

One of the most important notions in differential geometry, and in particular its application in physics, is that of a *connection*. For a general fiber bundle $\pi : E \to B$, one can find a number of equivalent definitions of a connection; one of the less common, but most useful approaches for its use in field theory is to define a connection as a section $\omega : E \to J^1(\pi)$ of the first jet bundle $\pi_{1,0} : J^1(\pi) \to E$, since it allows us to view a connection as a section of a fiber bundle, along with other physical fields [44]. Note, however, that the base space of this fiber bundle is the total space E of the original fiber bundle $\pi : E \to B$, since its value, in general, depends arbitrarily on the position along the fibers of E.

In order to be considered as a section of a fiber bundle over B, the aforementioned dependence must be further restricted. This is possible, for example, if $\pi : E \to B$ is a vector bundle, so that its fibers are vector spaces. In this case one finds that also $\pi_1 : J^1(\pi) \to B$ is a vector bundle, while $\pi_{1,0} : J^1(\pi) \to E$ is *not* a vector bundle, but an affine bundle. Then one commonly demands that ω is linear on every fiber of E, which is equivalent to demanding that $\omega : E \to J^1(\pi)$ is a vector bundle homomorphism covering the identity id_B on B, which in addition satisfies the property $\pi_{1,0} \circ \omega = \mathrm{id}_E$. Further using the fact that vector bundle homomorphisms covering the identity may equivalently be regarded as sections of a homomorphism bundle $\mathrm{Hom}(E, J^1(\pi))$ over the base manifold B, one may regard ω as a section of this homomorphism bundle, with the additional restriction arising from the condition involving $\pi_{1,0}$. Those elements of the homomorphism bundle, which satisfy this condition, do not form a vector bundle, but an affine bundle, modeled over the vector bundle

$$E \otimes E^* \otimes T^*B \,. \tag{2.1.7}$$

Given local coordinates (x^μ, y^a) on E as in the previous section, one may use this construction to introduce coordinates $(x^\mu, \omega^a{}_{b\mu})$ on the affine connection bundle, where $\omega^a{}_{b\mu}$ are usually called the *connection coefficients*. For a section $x^\mu \mapsto y^a(x^\mu)$, they give rise to the *covariant derivative*

$$\nabla_\mu y^a = \partial_\mu y^a + \omega^a{}_{b\mu} y^b \,. \tag{2.1.8}$$

If the bundle $\pi : E \to B$ is a natural vector bundle defined by a bundle functor $\mathscr{F}$, then also the connection bundle defined above is a natural affine bundle. In particular, in this thesis we make use of connections on the tangent bundle TB, defined by the tangent functor $\mathscr{T}$. In this case it is common to denote the coordinates on TB as $(x^\mu, \bar{x}^\mu)$, where we use the same indices, since the fiber coordinates follow canonically from those on the base manifold, and write the connection coefficients as $\Gamma^\mu{}_{\nu\rho}$. Here we remark that we prefer to retain the same order of the indices as in the general case, i.e., write the covariant derivative as

$$\nabla_\mu \bar{x}^\nu = \partial_\mu \bar{x}^\nu + \Gamma^\nu{}_{\rho\mu} \bar{x}^\rho \,, \tag{2.1.9}$$

so that they reflect the order of the factors in the tensor product bundle (2.1.7).

2.1.5 Diffeomorphisms and Lie derivative

In this thesis, our main focus is on diffeomorphisms, i.e., smooth, bijective maps, whose inverse is again smooth, and in particular on the case in which the domain and codomain of the diffeomorphism is given by the same manifold. Given a natural bundle $\pi : E \to B$, defined by a bundle functor $\mathscr{F}$, and a diffeomorphism $\psi : B \to B$, the lift $\Psi = \mathscr{F}(\psi)$ is a bundle isomorphism; the same holds true for its inverse. This allows an operation on sections as follows.

Definition 2.4 (Pullback)**.** Let B be a manifold, $\pi : E \to B$ a natural fiber bundle over B defined by a bundle functor $\mathscr{F}$ and $\psi : B \to B$ a diffeomorphism. For every section $\sigma : B \to E$, its *pullback* $\psi^*\sigma$ along ψ is defined as $\psi^*\sigma = \mathscr{F}(\psi^{-1}) \circ \sigma \circ \psi$.

Further, we will mostly be interested in sections which do not change under this operation.

Definition 2.5 (Invariant section)**.** Let B be a manifold, $\pi : E \to B$ a natural fiber bundle over B and $\psi : B \to B$ a diffeomorphism. A section $\sigma : B \to E$ is called *invariant* under ψ if and only if it agrees with its pullback, $\sigma = \psi^*\sigma$.

Finally, we will usually consider not only single diffeomorphisms, but a group of diffeomorphisms, where the group operation is given by map composition. Hence, we consider actions $\boldsymbol{\psi} : G \times B \to B$ of a group G, which will usually be a Lie group, such that for every $u \in G$, $\boldsymbol{\psi}_u : B \to B$ is a diffeomorphism. In the case $G = (\mathbb{R}, +)$ is the group of real numbers, with addition as the group multiplication, we call the group action $\boldsymbol{\psi}$ a one-parameter group. In the latter case, for every $b \in B$, the assignment $t \mapsto \boldsymbol{\psi}_t(b) = \gamma_b(t)$ with $t \in \mathbb{R}$ defines a curve on B, with $\gamma_b(0) = b$. It follows that its tangent vector $\dot{\gamma}_b(0) \in T_bB$ at $t = 0$ is a tangent vector at b. The assignment $X : b \mapsto X(b) = \dot{\gamma}_b(0)$ then defines a vector field on B, called the *generating vector field* of the one-parameter group $\boldsymbol{\psi}$, and it turns out that $\boldsymbol{\psi}$ is uniquely defined by X.

Given a one-parameter group $\boldsymbol{\psi} : \mathbb{R} \times B \to B$, generated by a vector field $X \in \operatorname{Vect} B$, acting on the base manifold B of a natural bundle $\pi : E \to B$ defined by a functor $\mathscr{F}$, one can obtain another construction, which is illustrated in figure 1. Given a section $\sigma : B \to E$, for every $b \in B$ one obtains a curve

$$t \mapsto (\boldsymbol{\psi}_t\sigma)(b) = (\psi^*\sigma = \mathscr{F}(\boldsymbol{\psi}_t^{-1}) \circ \sigma \circ \boldsymbol{\psi}_t)(b) \in E_b \tag{2.1.10}$$

on the fiber $E_b = \pi^{-1}(b) \subset E$ over B. Its tangent vector at $t = 0$ is therefore a vertical tangent vector $v \in V_{\sigma(b)}E$ at $\sigma(b) \in E$ where the vertical tangent bundle $VE = \ker \pi_* \subset TE$ consists of those tangent vectors $v \in TE$ which are tangent to the fibers E_b.

In the case that $\pi : E \to B$ is a natural affine bundle, modeled over a vector bundle $\vec{\pi} : \vec{E} \to B$, there exists a canonical identification of the vertical tangent space $V_{\sigma(b)}E$ with the fiber $\vec{E}_b$ of the underlying vector bundle. The tangent vector v is thus identified with the derivative

$$\left.\frac{\mathrm{d}(\boldsymbol{\psi}_t^*\sigma)(b)}{\mathrm{d}t}\right|_{t=0} = \lim_{t\to 0} \frac{(\boldsymbol{\psi}_t^*\sigma)(b) - \sigma(b)}{t} \in \vec{E}_b\,, \tag{2.1.11}$$

which is well defined, since $(\boldsymbol{\psi}_t^*\sigma)(b) \in E_b$ lies in the affine space E_b. The assignment of this element to every $b \in B$ defines a section of the vector bundle $\vec{\pi} : \vec{E} \to B$, which is defined as follows [166]:

Definition 2.6 (Lie derivative)**.** Let B be a manifold, $\pi : E \to B$ a natural affine bundle over B and $\boldsymbol{\psi} : \mathbb{R} \times B \to B$ a one-parameter group of diffeomorphisms generated by a vector field X on B. For every section $\sigma : B \to E$, its *Lie derivative* is the section $\mathcal{L}_X\sigma : B \to \vec{E}$ given by

$$\mathcal{L}_X\sigma = \lim_{t\to 0} \frac{\boldsymbol{\psi}_t^*\sigma - \sigma}{t}\,. \tag{2.1.12}$$

Most commonly, the Lie derivative is encountered in the case of vector bundles, and in particular the tensor bundles T^r_sB of tensors of rank (r, s), formed by a tensor product or r copies of the tangent bundle $E = TB$ and s copies of its dual $E^* = T^*B$, in which case it yields a section of the same bundle T^r_sB. However, it is less well-known that the Lie derivative can be defined also on affine bundles which are not vector bundles, and so in particular for affine connections, which will become relevant in our definitions in section 3.

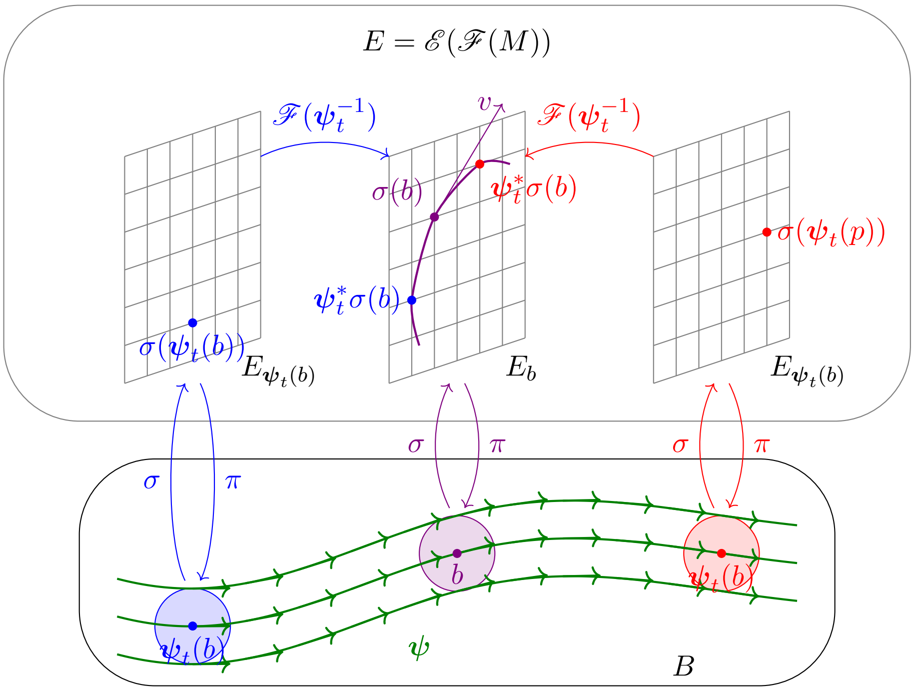

Figure 1: Illustration of the Lie derivative.

2.2 Particular geometries

We now introduce a few particular geometries, which are commonly used in the description of gravity theories, and which will further be used in this thesis. The most general class of geometries we consider is that of Cartan geometry, which we discuss in section 2.2.1. The remaining classes of geometries can be understood is particular classes of Cartan geometries, possibly equipped with additional geometric objects. In particular, we will discuss metric-affine geometry in section 2.2.2, teleparallel geometry in section 2.2.3 and Finsler geometry in section 2.2.4.

Since the main aim of this thesis is to make use of geometry as a language for the formulation of gravity theories, we are in particular interested in describing the geometry of spacetime. Unless otherwise specified, we will denote the spacetime manifold by the letter M, and assume that it is a four-dimensional, Hausdorff, paracompact, orientable, smooth manifold.

2.2.1 Cartan geometry

In this thesis we make use of different notions of geometries, which may be considered as special cases of Cartan geometry, whose key idea is to describe the geometry imposed on a manifold as its deviation from that of a homogeneous space, or Klein geometry [155]. The latter is defined as the coset space G/H of a Lie group G and a closed subgroup $H \subset G$. Given a Klein geometry G/H, one proceeds to define a Cartan geometry modeled on G/H as a principal H-bundle $\pi : P \to B$ together with a $\mathfrak{g}$-valued 1-form $\mathbf{A} \in \Omega^1(P, \mathfrak{g})$ on P, called the Cartan connection, which must satisfy the following conditions:

1. For each $p \in P$, $\mathbf{A}_p = \mathbf{A}|_{T_pP} : T_pP \to \mathfrak{g}$ is a linear isomorphism.

2. $\mathbf{A}$ is H-equivariant: $(R_h)^*\mathbf{A} = \mathrm{Ad}(h^{-1}) \circ \mathbf{A}$ for all $h \in H$, where $R : P \times H \to P$ denotes the right translation on the principal H-bundle P.

3. $\mathbf{A}(\tilde{h}) = h$ for all $h \in \mathfrak{h}$, where $\tilde{h}$ denotes the fundamental vector field of h, i.e., the corresponding generating vector field of right translations.

From the first condition follows in particular that $\dim P = \dim G$, since otherwise no such isomorphism could exist. Further taking into account that the dimension of the fibers of P is $\dim H$, it follows that the dimension of the base manifold is given by $\dim B = \dim G/H = \dim G - \dim H$.

An object of particular relevance on Cartan geometries is the Cartan curvature $\mathbf{F} \in \Omega^2(P, \mathfrak{g})$, which is defined as

$$\mathbf{F} = \mathrm{D}\mathbf{A} = \mathrm{d}\mathbf{A} + \frac{1}{2}[\mathbf{A} \wedge \mathbf{A}]\,, \tag{2.2.1}$$

where the notation $[\mathbf{A} \wedge \mathbf{A}]$ is used to indicate that the exterior product $\wedge$ acts on the differential form factors in the Cartan connection, while the Lie bracket $[,]$ acts on the Lie algebra factors.

In this thesis we consider in particular Cartan geometries which satisfy two additional conditions on their model Klein geometries:

1. A Cartan geometry with model Klein geometry G/H is called *first-order* Cartan geometry if the quotient representation of H on $\mathfrak{g}/\mathfrak{h}$ is faithful. Otherwise, it is called *higher-order* Cartan geometry.

2. A Cartan geometry with model Klein geometry G/H is called *reductive* if the Lie algebra $\mathfrak{g}$ allows a decomposition of the form $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{z}$ into subrepresentations of the adjoint representation of H.

First-order reductive Cartan geometries enjoy the property that their principal bundle P can be identified with a subbundle of the frame bundle $\mathrm{GL}(B)$, defined as

$$\mathrm{GL}(B) = \bigcup_{x \in B} \{\text{linear bijections } f : \mathbb{R}^n \to T_xB\}\,, \tag{2.2.2}$$

From this definition one easily constructs coordinates on $\mathrm{GL}(B)$ from coordinates on B, by writing a frame, i.e., a linear bijection $f : \mathbb{R}^n \to T_xB$ as

$$f : v = v^A \mathcal{Z}_A \mapsto v^A f_A{}^\mu \partial_\mu\,, \tag{2.2.3}$$

where $(\mathcal{Z}_A)$ denotes the canonical basis of $\mathbb{R}^n$, and the matrix components $f_A{}^\mu$ serve as fiber coordinates on each fiber $P_x = \pi^{-1}(x) \subset P$. For the corresponding coordinate vector fields in $\mathrm{GL}(B)$, we write

$$\partial_\mu = \frac{\partial}{\partial x^\mu}\,, \quad \bar{\partial}^A{}_\mu = \frac{\partial}{\partial f_A{}^\mu}\,. \tag{2.2.4}$$

One further finds that the Cartan connection of first-order reductive Cartan geometries splits in the form $\mathbf{A} = \boldsymbol{\omega} + \mathbf{e}$ with $\boldsymbol{\omega} \in \Omega^1(P, \mathfrak{h})$ and $\mathbf{e} \in \Omega^1(P, \mathfrak{z})$, where the latter is the restriction of the so-called solder form

$$\mathbf{e} = f^{-1}{}^A{}_\mu \mathcal{Z}_A \mathrm{d}x^\mu \tag{2.2.5}$$

to P. On a vector $w \in T_f\mathrm{GL}(B)$ it acts as

$$\mathbf{e}(w) = f^{-1}(\pi_*(w))\,, \tag{2.2.6}$$

by first pushing w to a tangent vector $\pi_* \in T_{\pi(f)}B$ at $\pi(f) \in B$, and then using the inverse frame $f^{-1} : T_{\pi(f)}B \to \mathfrak{z}$. In the following we will always assume that this identification for P and $\mathbf{A}$ is made. It then follows that $\boldsymbol{\omega}$ is a principal Ehresmann connection on P, so that there is a one-to-one correspondence between first-order Cartan geometries and Ehresmann connections on subbundles of the frame bundle.

2.2.2 Metric-affine geometry

A large class of geometries which are used in the description of gravity theories can be subsumed as special cases of metric-affine geometries [53]. For a general metric-affine geometry, the fundamental fields defined on the spacetime manifold M are a metric $g_{\mu\nu}$ and an independent affine connection with coefficients $\Gamma^\mu{}_{\nu\rho}$. The connection is characterized by two tensors, which are its curvature,

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu\Gamma^\rho{}_{\sigma\nu} - \partial_\nu\Gamma^\rho{}_{\sigma\mu} + \Gamma^\rho{}_{\lambda\mu}\Gamma^\lambda{}_{\sigma\nu} - \Gamma^\rho{}_{\lambda\nu}\Gamma^\lambda{}_{\sigma\mu} , \tag{2.2.7}$$

as well as the torsion

$$T^\rho{}_{\mu\nu} = \Gamma^\rho{}_{\nu\mu} - \Gamma^\rho{}_{\mu\nu} . \tag{2.2.8}$$

In addition, together with the metric it gives rise to the nonmetricity

$$Q_{\rho\mu\nu} = \nabla_\rho g_{\mu\nu} . \tag{2.2.9}$$

The presence of a metric further allows a unique decomposition of the affine connection in the form

$$\Gamma^\mu{}_{\nu\rho} = \overset{\circ}{\Gamma}{}^\mu{}_{\nu\rho} + K^\mu{}_{\nu\rho} + L^\mu{}_{\nu\rho} , \tag{2.2.10}$$

where $\overset{\circ}{\Gamma}{}^\mu{}_{\nu\rho}$ is the Levi-Civita connection of the metric $g_{\mu\nu}$, given by the usual formula

$$\overset{\circ}{\Gamma}{}^\rho{}_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}) , \tag{2.2.11}$$

$K^\mu{}_{\nu\rho}$ is the contortion

$$K^\mu{}_{\nu\rho} = \frac{1}{2}\left(T_\nu{}^\mu{}_\rho + T_\rho{}^\mu{}_\nu - T^\mu{}_{\nu\rho}\right) \tag{2.2.12}$$

and $L^\mu{}_{\nu\rho}$ is the disformation

$$L^\mu{}_{\nu\rho} = \frac{1}{2}\left(Q^\mu{}_{\nu\rho} - Q_\nu{}^\mu{}_\rho - Q_\rho{}^\mu{}_\nu\right) . \tag{2.2.13}$$

Various special cases are used in the description of gravity theories, which are defined by the vanishing of one or several of the tensorial quantities introduced above.

Another viewpoint on metric-affine geometry is obtained from its relation to Cartan geometry, which we defined in the previous section. The basic ingredient to construct this relation is the choice of a principal bundle $\pi : P \to M$ and a model Klein geometry G/K. In this case $P = \mathrm{GL}(M)$ is the general linear frame bundle (2.2.2). For the model geometry G/H one chooses the general affine group $\mathrm{Aff}(n, \mathbb{R}) = \mathrm{GL}(n, \mathbb{R}) \ltimes \mathbb{R}^n$, with closed subgroup $H = \mathrm{GL}(n, \mathbb{R})$. Its Lie algebra $\mathfrak{g} = \mathfrak{aff}(n, \mathbb{R}) \cong \mathfrak{gl}(n, \mathbb{R}) \oplus \mathbb{R} = \mathfrak{h} \otimes \mathfrak{z}$ allows a split into subrepresentations of H, with the representation on $\mathfrak{z}$ being faithful, and so the model is first-order and reductive. Together with the basis elements $\mathcal{H}_A{}^B$ of $\mathfrak{gl}(n, \mathbb{R})$ one has the commutation relations

$$[\mathcal{H}_A{}^B, \mathcal{H}_C{}^D] = \delta^B_C\mathcal{H}_A{}^D - \delta^D_A\mathcal{H}_C{}^B , \quad [\mathcal{H}_A{}^B, \mathcal{Z}_C] = \delta^B_C\mathcal{Z}_A , \quad [\mathcal{Z}_A, \mathcal{Z}_B] = 0 . \tag{2.2.14}$$

With the help of this basis, one finds that the $\mathfrak{g}$-valued one-form $\mathbf{A} \in \Omega^1(P, \mathfrak{g})$ defined by

$$\mathbf{A} = \mathbf{e} + \boldsymbol{\omega} = f^{-1\,A}{}_{\mu}\left[\mathcal{Z}_A \mathrm{d}x^\mu + \mathcal{H}_A{}^B(\mathrm{d}f_B{}^\mu + f_B{}^\nu \Gamma^\mu{}_{\nu\rho}\mathrm{d}x^\rho)\right] \tag{2.2.15}$$

constitutes a Cartan connection, where

$$\boldsymbol{\omega} = f^{-1\,A}{}_{\mu}\mathcal{H}_A{}^B(\mathrm{d}f_B{}^\mu + f_B{}^\nu \Gamma^\mu{}_{\nu\rho}\mathrm{d}x^\rho) \tag{2.2.16}$$

is the principal Ehresmann connection corresponding to the affine connection. Its Cartan curvature $\mathbf{F} \in \Omega^2(P, \mathfrak{g})$ is given by

$$\mathbf{F} = \mathrm{D}\mathbf{A} = \frac{1}{2}\left(f^{-1\,A}{}_{\rho} T^\rho{}_{\mu\nu}\mathcal{Z}_A + f^{-1\,A}{}_{\rho} f^B{}_{\sigma} R^\rho{}_{\sigma\mu\nu}\mathcal{H}_A{}^B\right)\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \mathbf{T} + \mathbf{R}\,, \tag{2.2.17}$$

in terms of the curvature (2.2.7) and torsion (2.2.8) of the affine connection. Further, the metric $g_{\mu\nu}$ can be lifted to P to give rise to a field $\mathbf{g} \in \Omega^0(P, \mathfrak{z}^* \otimes \mathfrak{z}^*)$ given by

$$\mathbf{g} = f_A{}^\mu f_B{}^\nu g_{\mu\nu}\mathcal{Z}^A \otimes \mathcal{Z}^B\,, \tag{2.2.18}$$

whose covariant derivative is the nonmetricity $\mathbf{Q} \in \Omega^1(P, \mathfrak{z}^* \otimes \mathfrak{z}^*)$ given by

$$\mathbf{Q} = \mathrm{D}\mathbf{g} = f_A{}^\mu f_B{}^\nu Q_{\rho\mu\nu}\mathcal{Z}^A \otimes \mathcal{Z}^B\,\mathrm{d}x^\rho\,. \tag{2.2.19}$$

This identification allows to treat metric-affine geometry in the more general framework of Cartan geometry.

2.2.3 Teleparallel geometry

A number of special cases of the aforementioned metric-affine geometry, which are obtained by posing additional restrictions on the affine connection, are known as teleparallel geometries. One can distinguish three different subtypes of teleparallel geometries:

1. The common, defining property of these geometries is the vanishing curvature (2.2.7) of the connection, $R^\rho{}_{\sigma\mu\nu} \equiv 0$. If no further conditions are imposed on the connection, the geometry is said to be *general teleparallel*. It is characterized by its torsion (2.2.8) and nonmetricity (2.2.9).

2. If in addition to vanishing curvature also the nonmetricity is imposed to vanish, the resulting geometry is *metric teleparallel*. In this case the only non-vanishing tensor property is the torsion.

3. Finally, one may impose both the curvature and torsion to vanish, leaving only nonmetricity as non-vanishing tensor field. In this case the connection is symmetric, and the resulting geometry is therefore called *symmetric teleparallel*.

In this thesis, we will mostly encounter the metric teleparallel geometry. Instead of the metric-affine formulation in terms of a metric and a flat, metric-compatible affine connection, a different description is more common in this case. One of the fundamental fields in this description is a *coframe* or *tetrad*, i.e., a section $\theta : M \to \mathrm{GL}^*(M)$ of the coframe bundle

$$\mathrm{GL}^*(M) = \bigcup_{x \in M}\{\text{linear bijections } f^{-1} : T_xM \to \mathbb{R}^n\}\,, \tag{2.2.20}$$

which can be expressed using the coordinates $(x^\mu, f^{-1\,A}{}_\mu)$ on $\mathrm{GL}^*(M)$ as $\theta^A{}_\mu$. Its inverse is the *frame*, which is a section of the frame bundle, and whose coordinate form $e_A{}^\mu$ is uniquely defined from that of the coframe by

$$\theta^A{}_\mu e_A{}^\nu = \delta^\nu_\mu\,, \quad \theta^A{}_\mu e_B{}^\mu = \delta^A_B\,. \tag{2.2.21}$$

Together with the Minkowski metric $\eta_{AB} = \mathrm{diag}(-1,1,1,1)$, the tetrad defines a Lorentzian metric

$$g_{\mu\nu} = \eta_{AB}\theta^A{}_\mu\theta^B{}_\nu\,. \tag{2.2.22}$$

In the original, non-covariant formulation of teleparallel geometry, the tetrad is the only fundamental field, and it is also used to define the affine connection as the Weitzenböck connection

$$\overset{\bullet}{\Gamma}{}^\mu{}_{\nu\rho} = e_A{}^\mu\partial_\rho\theta^A{}_\nu\,, \tag{2.2.23}$$

where we introduced a bullet to distinguish the flat, metric-compatible connection and its related tensorial quantities from that given by the general affine connection. Indeed, one finds that the Weitzenböck connection is flat and compatible with the metric (2.2.22). Further, it turns out that the same metric and Weitzenböck connection are obtained from any other tetrad $\theta'^A{}_\mu$ which is related to $\theta^A{}_\mu$ by a global Lorentz transformation

$$\theta'^A{}_\mu = \Lambda^A{}_B\theta^B{}_\mu\,, \quad \partial_\mu\Lambda^A{}_B = 0\,. \tag{2.2.24}$$

However, in this thesis we make use of a different approach to teleparallel geometry, known as its covariant formulation [122, 86, 121]. In this formulation another fundamental field is introduced, called the *spin connection* $\overset{\bullet}{\omega}{}^A{}_{B\mu}$, on which one imposes the flatness condition,

$$\partial_\mu\overset{\bullet}{\omega}{}^A{}_{B\nu} - \partial_\nu\overset{\bullet}{\omega}{}^A{}_{B\mu} + \overset{\bullet}{\omega}{}^A{}_{C\mu}\overset{\bullet}{\omega}{}^C{}_{B\nu} - \overset{\bullet}{\omega}{}^A{}_{C\nu}\overset{\bullet}{\omega}{}^C{}_{B\mu} \equiv 0\,, \tag{2.2.25}$$

as well as the metricity condition

$$\eta_{AC}\overset{\bullet}{\omega}{}^C{}_{B\mu} + \eta_{BC}\overset{\bullet}{\omega}{}^C{}_{A\mu} \equiv 0\,. \tag{2.2.26}$$

These two conditions are in one-to-one correspondence with the flatness and metricity conditions on the affine connection defined by

$$\overset{\bullet}{\Gamma}{}^\mu{}_{\nu\rho} = e_A{}^\mu\left(\partial_\rho\theta^A{}_\nu + \overset{\bullet}{\omega}{}^A{}_{B\rho}\theta^B{}_\nu\right)\,, \tag{2.2.27}$$

and so they define the most general metric teleparallel geometry. The covariance of this formulation of teleparallel geometry becomes apparent in various relations. First, note that the global Lorentz invariance is enhanced to a local Lorentz invariance, $\partial_\mu\Lambda^A{}_B \neq 0$, which means that the metric (2.2.22) and affine connection (2.2.27) are invariant under a simultaneous local Lorentz transformation of the tetrad and the spin connection given by

$$\theta'^A{}_\mu = \Lambda^A{}_B\theta^B{}_\mu\,, \quad \overset{\bullet}{\omega}{}'^A{}_{B\mu} = \Lambda^A{}_C(\Lambda^{-1})^D{}_B\overset{\bullet}{\omega}{}^C{}_{D\mu} + \Lambda^A{}_C\partial_\mu(\Lambda^{-1})^C{}_B\,. \tag{2.2.28}$$

Further, the spin connection and affine connection together allow to construct a fully covariant derivative, which is covariant both under diffeomorphisms and under local Lorentz transformations, where the spin connection defines the coefficients of the Lorentz covariant derivative. Using this fully covariant derivative, the relation (2.2.27) between the affine and spin connection can be expressed through the so-called "tetrad postulate"

$$\partial_\mu\theta^A{}_\nu + \overset{\bullet}{\omega}{}^A{}_{B\mu}\theta^B{}_\nu - \overset{\bullet}{\Gamma}{}^\rho{}_{\nu\mu}\theta^A{}_\rho = 0\,, \tag{2.2.29}$$

which states that the tetrad is covariantly constant with respect to the fully covariant derivative.

Since teleparallel geometries are a special case of metric-affine geometries, it is in particular possible to express them in terms Cartan geometry, following the same line of thought as in the general case shown in section 2.2.2. It follows from the flatness condition on the teleparallel connection that the resulting curvature $\mathbf{R}$ vanishes, while the metricity condition implies vanishing nonmetricity $\mathbf{Q}$. From the latter follows that the restriction of the Cartan connection $\mathbf{A}$ to the metric frame bundle

$$\mathrm{SO}(M,g) = \{f \in \mathrm{GL}(M), f_A{}^\mu f_B{}^\nu g_{\mu\nu} = \eta_{AB}\} \tag{2.2.30}$$

can be written as

$$\overset{\bullet}{\mathbf{A}} = f^{-1}{}^A{}_\mu \left[\mathcal{Z}_A \mathrm{d}x^\mu + \frac{1}{2}\tilde{\mathcal{H}}_A{}^B(\mathrm{d}f_B{}^\mu + f_B{}^\nu \overset{\bullet}{\Gamma}{}^\mu{}_{\nu\rho}\mathrm{d}x^\rho)\right], \tag{2.2.31}$$

with

$$\tilde{\mathcal{H}}_A{}^B = \mathcal{H}_A{}^B - \eta_{AC}\eta^{BD}\mathcal{H}_D{}^C = -\eta_{AC}\eta^{BD}\tilde{\mathcal{H}}_D{}^C, \tag{2.2.32}$$

due to the antisymmetry of the spin connection. It thus takes values in the Poincaré algebra $\mathfrak{g} = \mathfrak{iso}(1,3)$, or any of its deformations $\mathfrak{g} = \mathfrak{so}(1,4)$ or $\mathfrak{g} = \mathfrak{so}(2,3)$, whose generators satisfy the commutation relations

$$\begin{gathered}[][\tilde{\mathcal{H}}_A{}^B, \tilde{\mathcal{H}}_C{}^D] = \delta^B_C\tilde{\mathcal{H}}_A{}^D - \delta^D_A\tilde{\mathcal{H}}_C{}^B + \eta_{AC}\eta^{DE}\tilde{\mathcal{H}}_E{}^B - \eta^{BD}\eta_{CE}\tilde{\mathcal{H}}_A{}^E, \\ [\tilde{\mathcal{H}}_A{}^B, \mathcal{Z}_C] = \delta^B_C\mathcal{Z}_A - \eta_{AC}\eta^{BD}\mathcal{Z}_D, \quad [\mathcal{Z}_A, \mathcal{Z}_B] = \Lambda\eta_{AC}\tilde{\mathcal{H}}_B{}^C,\end{gathered} \tag{2.2.33}$$

where Λ depends on the choice of the algebra used in the construction of the orthogonal Cartan geometry [65]. This can be related to the usual description of the teleparallel geometry in terms of the tetrad and spin connection as follows. Using the frame $e_A{}^\mu$, which is a section of the frame bundle $\mathrm{GL}(M)$, and in particular of the orthonormal frame bundle $\mathrm{SO}(M,g)$ due to the relation (2.2.22), one has the pullback

$$e^*\overset{\bullet}{\mathbf{A}} = \theta^A{}_\mu \left[\mathcal{Z}_A\mathrm{d}x^\mu + \mathcal{H}_A{}^B(\partial_\rho e_B{}^\mu + e_B{}^\nu\overset{\bullet}{\Gamma}{}^\mu{}_{\nu\rho})\mathrm{d}x^\rho\right] = \theta^A\mathcal{Z}_A + \overset{\bullet}{\omega}{}^A{}_B\mathcal{H}_A{}^B, \tag{2.2.34}$$

and so one immediately obtains the tetrad and the spin connection. Local Lorentz transformations then simply relate different choices of the tetrad to each other.

2.2.4 Finsler geometry

Another class of geometries, which is used in the description of gravity theories, and which differs significantly from the aforementioned geometries, is given by Finsler geometry. Its most distinguishing property is the fact that it is not modeled by tensor fields and connections on the spacetime manifold M, but by a non-negative length function $F : TM \to \mathbb{R}^+_0$ on the total space TM of its tangent bundle $\tau : TM \to M$. In order to simplify working with objects on the tangent bundle, it is usually convenient to introduce *induced coordinates* $(x^\mu, \bar{x}^\mu)$ on TM from a given set of coordinates (x^μ) on M, where $(x^\mu, \bar{x}^\mu)$ denotes the tangent vector

$$\bar{x}^\mu\partial_\mu = \bar{x}^\mu\frac{\partial}{\partial x^\mu} \in T_xM, \tag{2.2.35}$$

where $x \in M$ is the point with coordinates (x^μ) on the base manifold, and to write the corresponding coordinate vector fields on TM as

$$\partial_\mu = \frac{\partial}{\partial x^\mu}, \quad \bar{\partial}_\mu = \frac{\partial}{\partial \bar{x}^\mu}. \tag{2.2.36}$$

In its original formulation [36], Finsler geometry was constructed as a generalization of Riemannian geometry (and hence with a metric of positive signature). In order to apply it to describe the geometry of spacetime, and hence allow for a Lorentzian signature, various generalizations have been developed [8, 141, 125, 108, 93]. For convenience and simplicity, we will use the following definition here [58, 59], which is not the most general and excludes several physically motivated examples, but serves the purpose of this thesis, and can easily be extended.

Definition 2.7 (Finsler spacetime). A *Finsler spacetime* (M, L, F) of dimension n is a n-dimensional, connected, Hausdorff, paracompact, smooth manifold M equipped with continuous real functions L, F on the tangent bundle TM which has the following properties:

1. L is smooth on the tangent bundle without the zero section $TM \setminus \{0\}$.
2. L is positively homogeneous of real degree $h \geq 2$ with respect to the fiber coordinates of TM,
$$L(x, \lambda\bar{x}) = \lambda^h L(x, \bar{x}) \quad \forall \lambda > 0\,, \tag{2.2.37}$$
and defines the Finsler function F via $F(x, \bar{x}) = |L(x, \bar{x})|^{\frac{1}{h}}$.
3. L is reversible: $|L(x, -\bar{x})| = |L(x, \bar{x})|$.
4. The Hessian
$$g^L_{\mu\nu}(x, \bar{x}) = \frac{1}{2}\bar{\partial}_\mu\bar{\partial}_\nu L(x, \bar{x}) \tag{2.2.38}$$
of L with respect to the fiber coordinates is non-degenerate on $TM \setminus X$, where $X \subset TM$ has measure zero and does not contain the null set $\{(x, \bar{x}) \in TM | L(x, \bar{x}) = 0\}$.
5. The unit timelike condition holds, i.e., for all $x \in M$ the set
$$\Omega_x = \left\{\bar{x} \in T_xM \,\middle|\, |L(x, \bar{x})| = 1, g^L_{\mu\nu}(x, \bar{x}) \text{ has signature } (\epsilon, -\epsilon, \ldots, -\epsilon), \epsilon = \frac{L(x, \bar{x})}{|L(x, \bar{x})|}\right\} \tag{2.2.39}$$
contains a non-empty closed connected component $S_x \subseteq \Omega_x \subset T_xM$.

A number of additional geometric objects on the tangent bundle TM are defined by the Finsler geometry outlined above. Among the most important ones, which we encounter in this thesis, are the Finsler metric

$$g^F_{\mu\nu}(x, \bar{x}) = \frac{1}{2}\bar{\partial}_\mu\bar{\partial}_\nu F^2(x, \bar{x}) \tag{2.2.40}$$

and the non-linear connection, which can be described by a split $TTM = VTM \oplus HTM$ of the double tangent bundle into vertical and horizontal bundles [21]. The vertical bundle VTM is canonically defined and spanned by the vector fields $\bar{\partial}_\mu$ in the induced coordinates, while the horizontal bundle HTM is spanned by the Berwald basis vector fields

$$\delta_\mu = \partial_\mu - N^\nu{}_\mu\bar{\partial}_\nu\,, \tag{2.2.41}$$

where the connection coefficients are derived from the Finsler geometry function as

$$N^\mu{}_\nu = \frac{1}{4}\bar{\partial}_\nu\left[g^{F\,\mu\rho}(\bar{x}^\sigma\partial_\sigma\bar{\partial}_\rho F^2 - \partial_\rho F^2)\right]\,; \tag{2.2.42}$$

the corresponding dual split of the cotangent bundle is given by the dual basis

$$\mathrm{d}x^\mu\,, \quad \delta\bar{x}^\mu = \mathrm{d}\bar{x}^\mu + N^\mu{}_\nu \mathrm{d}x^\nu\,. \tag{2.2.43}$$

The Berwald basis given above is instrumental in the definition of an important class of objects commonly encountered in Finsler geometry, known as d-tensors, which were introduced in [132]. Conventionally, their definition relies on the non-linear connection, defined via the split $TTM = VTM \oplus HTM$ of the tangent bundle. However, for the discussion relevant to this thesis a more convenient definition can be obtained from the so-called pullback bundle approach [3, 157]. Its central definition is that of the pullback bundle $\pi : PM \to TM$, where $PM = TM \times_M TM$ is a fibered product and π is the projection onto the first factor of this product. It is a vector bundle over TM, whose fibers are isomorphic to the fibers of TM, and whose dual is given by $P^*M = TM \times_M T^*M$. Hence, for each $v \in TM$, a basis of P_vM is given by a pullback of the basis vectors $\partial_\mu \in T_{\tau(v)}M$, while a basis of P^*_vM is constituted by $\mathrm{d}x^\mu \in T^*_{\tau(v)}M$. It follows that a section $A \in \Gamma(P^r_sM)$ of the tensor bundle

$$P^r_sM = \underbrace{PM \otimes \cdots \otimes PM}_{r\text{ times}} \otimes \underbrace{P^*M \otimes \cdots \otimes P^*M}_{s\text{ times}}\,, \tag{2.2.44}$$

which is again a vector bundle over TM, takes the same component form $A^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\nu_s}$ in coordinates as a tensor field, i.e., a section of T^r_sM, with the only difference that its components are functions on TM instead of M. We call such a section of the corresponding bundle $\pi^r_s : P^r_sM \to TM$ a *d-tensor field* of rank (r, s). One finds that the Finsler metric $g^F_{\mu\nu}$ and the Hessian $g^L_{\mu\nu}$ form the components of d-tensor fields. Moreover, the Finsler geometry defines a number of connections on the pullback bundle and its tensor bundles. Here we restrict ourselves to the Cartan linear connection. Given a vector field $X = X^\mu\delta_\mu + \bar{X}^\mu\bar{\partial}_\mu \in \mathrm{Vect}(TM)$, where we express the components in the Berwald basis, and a d-vector field $Y \in \Gamma(PM)$, the Cartan linear connection defines the covariant derivative

$$(\nabla_XY)^\mu = X^\nu(\delta_\nu Y^\mu + F^\mu{}_{\rho\nu}Y^\rho) + \bar{X}^\nu(\bar{\partial}_\nu Y^\mu + C^\mu{}_{\rho\nu}Y^\rho)\,, \tag{2.2.45}$$

where the connection coefficients are given by

$$F^\mu{}_{\nu\rho} = \frac{1}{2}g^{F\,\mu\sigma}(\delta_\mu g^F_{\rho\sigma} + \delta_\rho g^F_{\nu\sigma} - \delta_\sigma g^F_{\nu\rho})\,, \quad C^\mu{}_{\nu\rho} = \frac{1}{2}g^{F\,\mu\sigma}(\bar{\partial}_\mu g^F_{\rho\sigma} + \bar{\partial}_\rho g^F_{\nu\sigma} - \bar{\partial}_\sigma g^F_{\nu\rho})\,, \tag{2.2.46}$$

and thus defined from the Finsler metric similarly to the Christoffel symbols in Riemannian geometry.

Finally, we remark that also Finsler spacetimes as defined above can be described using Cartan geometry [58]; this will be used for the definition of symmetries in section 3.1. However, different from the metric-affine and teleparallel geometries discussed before, the base manifold of this Cartan geometry is not the spacetime manifold M, but the so-called observer space [48]

$$O = \bigcup_{x\in M} S_x\,, \tag{2.2.47}$$

which is a 7-dimensional manifold composed from the unit timelike vectors S_x at each spacetime point $x \in S_x$. The principal bundle on which the Cartan connection is defined has the structure group $K = \mathrm{SO}(3)$ and is defined by

$$P = \left\{(x, f) \in \mathrm{GL}(M) \,|\, f_0 \in O \text{ and } g^F_{\mu\nu}(x, f_0)f_A{}^\mu f_B{}^\nu = -\eta_{AB}\right\}\,. \tag{2.2.48}$$

This structure group is a closed subgroup of the Lorentz group $H = \mathrm{SO}(1,3)$, which in turn is a closed subgroup of any of the groups $\mathrm{ISO}(1,3)$, $\mathrm{SO}(1,4)$ and $\mathrm{SO}(2,3)$, and one may choose any of them to construct a Cartan geometry modeled on the Klein geometry G/K. This Klein geometry is reductive, as it allows a split of the Lie algebra $\mathfrak{g}$ of G in the form

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{y} \oplus \vec{\mathfrak{z}} \oplus \mathfrak{z}^0 \tag{2.2.49}$$

into subrepresentations of K, which are interpreted as spatial rotations, boosts, spatial translations and temporal translations. The basis elements of these vector spaces are related to the generators (2.2.32) of the Lorentz algebra $\mathfrak{k} \oplus \mathfrak{y}$ as

$$\mathcal{K}_a{}^b = \tilde{\mathcal{H}}_a{}^b\,, \quad \mathcal{Y}_a = \tilde{\mathcal{H}}_a{}^0\,, \tag{2.2.50}$$

together with the translation generators $\mathcal{Z}_A$, which decompose into spatial components $\mathcal{Z}_a$ and a time component $\mathcal{Z}_0$. Their commutation relations follow from a decomposition of the relations (2.2.33) and read

$$\begin{gathered}
[\mathcal{K}_a{}^b, \mathcal{K}_c{}^d] = \delta^b_c\mathcal{K}_a{}^d - \delta^d_a\mathcal{K}_c{}^b + \delta_{ac}\delta^{de}\mathcal{K}_e{}^b - \delta^{bd}\delta_{ce}\mathcal{K}_a{}^e\,, \quad [\mathcal{K}_a{}^b, \mathcal{Z}_0] = 0\,, \\
[\mathcal{K}_a{}^b, \mathcal{Y}_c] = \delta^b_c\mathcal{Y}_a - \delta_{ac}\delta^{bd}\mathcal{Y}_d\,, \quad [\mathcal{K}_a{}^b, \mathcal{Z}_c] = \delta^b_c\mathcal{Z}_a - \delta_{ac}\delta^{bd}\mathcal{Z}_d\,, \quad [\mathcal{Y}_a, \mathcal{Y}_b] = -\delta_{ac}\mathcal{K}_b{}^c\,, \\
[\mathcal{Y}_a, \mathcal{Z}_b] = \delta_{ab}\mathcal{Z}_0\,, \quad [\mathcal{Y}_a, \mathcal{Z}_0] = \mathcal{Z}_a\,, \quad [\mathcal{Z}_a, \mathcal{Z}_b] = \Lambda\delta_{ac}\mathcal{K}_b{}^c\,, \quad [\mathcal{Z}_a, \mathcal{Z}_0] = \Lambda\mathcal{Y}_a\,.
\end{gathered} \tag{2.2.51}$$

Also note that the restriction of the adjoint representation of $K \subset G$ on $\mathfrak{y} \oplus \vec{\mathfrak{z}} \oplus \mathfrak{z}^0$ is faithful, so that this is a first order Klein geometry. One finds that the Finsler spacetime geometry defined above induces a Cartan geometry on the bundle $\pi : P \to O$, whose Cartan connection reads [58]

$$\mathbf{A} = f^{-1}{}^A{}_\mu\mathcal{Z}_A\mathrm{d}x^\mu + f^{-1}{}^a{}_\mu\left\{\mathcal{Y}_a\delta\bar{x}^\mu + \frac{1}{2}\mathcal{K}_a{}^b\left[\mathrm{d}f_b{}^\mu + f_b{}^\nu(F^\mu{}_{\nu\rho}\mathrm{d}x^\rho + C^\mu{}_{\nu\rho}\delta\bar{x}^\rho)\right]\right\}\,. \tag{2.2.52}$$

Hence, Finsler spacetimes may be described as first-order reductive Cartan geometries on their observer space. Even more remarkable is the fact that the Cartan geometry comprising of the observer bundle $\pi : P \to O$ and the Cartan connection $\mathbf{A}$ fully encode the original Finsler spacetime geometry, so that it is possible to reconstruct both the underlying spacetime manifold M and the geometry function L. This circumstance allows studying Finsler spacetimes fully from the observer perspective, in terms of their induced observer space Cartan geometry. Moreover, one may also consider more general observer space Cartan geometries, for which no underlying spacetime manifold exists, and which can nevertheless be interpreted as modeling the dynamics on a space of local observers [48]. The necessary ingredient for this interpretation is the split (2.2.49) of the Lie algebra of the structure group G, through which the Cartan connection $\mathbf{A}$ induces a corresponding split

$$TP = RP \oplus BP \oplus \vec{H}P \oplus H^0P \tag{2.2.53}$$

of the tangent bundle TP. Here the first part RP is given by rotations, i.e., vertical tangent vectors to the bundle $\pi : P \to O$, so that $RP = \ker \pi_*$. Since the split (2.2.49) is invariant under the adjoint action of K, the projection $\pi_* : TP \to TO$ induces a corresponding split

$$TO = VO \oplus \vec{H}O \oplus H^0O \tag{2.2.54}$$

of TO. Here $BP = \pi_*^{-1}(VO)$ corresponds to infinitesimal boosts of an observer, while $\vec{H}P = \pi_*^{-1}(\vec{H}O)$ and $H^0P = \pi_*^{-1}(H^0O)$ describe spatial and temporal translations, respectively.

2.2.5 Geometry of scalar fields

The arguably most simple type of field theories discussed in physics is that of scalar field theories, where the dynamical field is conventionally as a real or complex function on the spacetime manifold M, hence $\phi : M \to \mathbb{R}$ or $\phi : M \to \mathbb{C}$, depending on the theory under consideration. Here we will restrict our attention to real scalar fields, since any complex scalar field can be represented by a pair of real fields. In the case of multiple scalar fields, the most intuitive and natural approach suggests to group these fields into a multiplet $\boldsymbol{\phi} : M \to \mathbb{R}^n$, where n denotes the number of scalar fields. This suggests that the underlying geometric structure of a multi-scalar field theory is that of a trivial bundle $M \times \mathbb{R}^n$, so that scalar field multiplets are sections of this bundle.

While the aforementioned approach conveys the idea that the value of each scalar field is a numeric quantity that can be associated a physical meaning, it is not the most general approach, and does not capture the full spirit of a geometric field theory. From the latter point of view, it is more natural to assume that the codomain of a scalar field multiplet $\boldsymbol{\phi}$ is actually an arbitrary field manifold F, possibly equipped with additional structure, so that the realm of multi-scalar field theories becomes the trivial bundle $M \times F$. In this approach numerical values for scalar fields arise merely through the choice of coordinates on F, or by considering further, real or complex functions defined on F, which are then evaluated at the point specified by the scalar field multiplet. In the context of multi-scalar extensions to gravity theories, where it is commonplace to consider redefinitions of scalar fields, this approach has the advantage that it gives this procedure a clear and simple geometric interpretation as a coordinate transformation of the field space F, and allows to define physical quantities in a coordinate-independent fashion. Note that this interpretation is by no means new; it is most commonly encountered in the context of so-called sigma models [154], where the field space F is usually taken to be a Lie group or homogeneous space, and thus naturally carries a transitive Lie group action.

2.3 Transformations of geometries and symmetries

Symmetry transformations and the invariance under such transformations have numerous applications in the mathematical description of physical theories. In this thesis we focus on two classes of symmetry transformations, which we will discuss in the course of this thesis together with their application to gravitational theories and their solutions. In section 2.3.1, we discuss the invariance of physical field configurations, which we model as sections of a fiber bundle, under symmetry transformations acting on the base space of this bundle; the latter implies that such transformations are infinitesimally generated by vector fields on the base space. The essentially opposite case is discussed in section 2.3.3, where we discuss the action of transformation groups on the space of field configurations which are generated by vector fields which are tangent to the fibers of a suitable bundle, and the invariance of Lagrangian field theories under such transformations.

2.3.1 Sections with base space symmetries

One of the most common task in the area of field theory and in particular the theoretical description of gravity is the derivation of solutions of a given theory, modeled by sections $\sigma : B \to E$ of a natural fiber bundle $\pi : E \to B$ with bundle functor $\mathscr{F}$, which are invariant under the action $\boldsymbol{\psi} : G \times B \to B$ of a given symmetry group G on the base space B. In the most frequently encountered cases in gravity theory, the base space is the four-dimensional spacetime manifold $B = M$, while frequently encountered transformation groups include the two-dimensional Euclidean group ISO(2) in the description of plane waves, the rotation group SO(3) in the case of spherical symmetry (possible extended to O(3) to include also reflections) and various groups describing spatial isotropy and homogeneity in the case of cosmology. As discussed in section 2.1.5, such the action ψ introduce an action

$$\begin{array}{rcrcl} \psi^* & : & G \times \Gamma(\pi) & \to & \Gamma(\pi) \\ & & (u, \sigma) & \mapsto & \psi_u^* \sigma \end{array} \tag{2.3.1}$$

on the space $\Gamma(\pi)$ of sections through the pullback.

In the usually encountered case in gravity theory, in which $\pi : E \to B$ is an affine bundle modeled on a vector bundle $\vec{\pi} : \vec{E} \to B$, such as a connection bundle, or already a vector bundle, it is often more convenient to consider infinitesimal symmetries instead, which are generated by the Lie algebra $\mathfrak{g}$ of G. Denoting by $X_\xi \in \mathrm{Vect}(B)$ the fundamental vector field of $\xi \in \mathfrak{g}$, i.e., the generating vector field of the action of the one-parameter subgroup $t \mapsto \exp(-t\xi)$ of G on B, the action on sections is given by

$$\begin{array}{rcrcl} \mathrm{d}\psi^* & : & \mathfrak{g} \times \Gamma(\pi) & \to & \Gamma(\vec{\pi}) \\ & & (\xi, \sigma) & \mapsto & \mathcal{L}_{X_\xi} \sigma \end{array} \tag{2.3.2}$$

in terms of the Lie derivative.

In the case of natural bundles, the finite and infinitesimal actions on the space of sections are canonically defined as shown in section 2.1.5. However, also more general situations are encountered in gravity theory, such as in the case of Cartan geometry or Finsler geometry, which require an extension to the notions given above. We provide such an extension to a number of relevant geometries in section 3.

Most often one is interested in field configurations σ which obey the considered symmetry, i.e., which are invariant under the action of the symmetry group. In the case of a finite Lie group action ψ of a group G, this means that $\psi_u^* \sigma = \sigma$ for every $u \in G$. Taking into account the definition (2.1.12) of the Lie derivative, it follows that the corresponding notion of invariance for infinitesimal transformations generated by the action of a Lie algebra $\mathfrak{g}$ is given by the condition $\mathcal{L}_{X_\xi} \sigma = 0$ for all $\xi \in \mathfrak{g}$, where the right hand side is understood as the canonical zero section $0 \in \Gamma(\vec{\pi})$ of the vector bundle $\vec{\pi} : \vec{E} \to B$. For a number of geometries used in gravity theories and transformation groups, we derive the most general class of such invariant sections in section 4.

While the aforementioned case of sections which are invariant under the action of the symmetry group is the most simple one, one frequently encounters also more general cases. For instance, if $\pi : E \to B$ is a vector bundle, also $\Gamma(\pi)$ inherits the structure of a vector space. In this case the bundle functor $\mathscr{F}$ lifts the group action ψ on B to vector bundle morphisms, so that the group action (2.3.1) acts linearly on the space of sections, hence giving a representation of the group G on $\Gamma(\pi)$. A natural question arising in this case is whether the representation can be decomposed into irreducible representations, and whether these are of finite dimension. This question motivates the study of invariant subspaces of $\Gamma(\pi)$. In particular, one-dimensional invariant subspaces on which G acts trivially are spanned by invariant sections. A non-trivial example is studied in section 5.3.

2.3.2 Perturbations of symmetric sections

Another possibility is to study the action of the whole diffeomorphism group $\mathrm{Diff}(B)$ on B, or in the infinitesimal case the Lie algebra $\mathfrak{diff}(B)$, which can be identified with the vector fields $\mathrm{Vect}(B)$ on B. The latter can be seen to act on the space of linear perturbations around a fixed background section $\bar{\sigma} : B \to E$, which can be identified with the pullback bundle $\bar{\sigma}^*VE$ of the vertical tangent bundle of E to B. In the most simple case that $\pi : E \to B$ is an affine bundle, this pullback bundle is identified with the underlying vector bundle $\vec{\pi} : \vec{E} \to B$. A linear perturbation is then given by a section $\delta\sigma : \vec{E} \to B$. The action of an infinitesimal diffeomorphism, expressed by a vector field $X \in \mathrm{Vect}(B)$, on this space of linear perturbations is given by

$$(\delta\sigma, X) \mapsto \delta\sigma - \pounds_X\bar{\sigma} \,. \tag{2.3.3}$$

Note, however, that this is not a Lie algebra action; it does not commute with the Lie bracket of vector fields. Since the vector fields act as translations in the space of linear perturbations, they generate a vector subspace. Taking the quotient of the vector space of all linear perturbations by this subspace leads to the space of gauge-invariant linear perturbations; this space is commonly used as the "physical" space of perturbations, with the subspaces generated by translations as the "pure gauge" part.

Conventionally, the background section is chosen to be a section of a natural bundle, which is symmetric under the action of a Lie group G on the base manifold B, as discussed in the previous section. In this case the group action of G on $\Gamma(\pi)$ induces an action on the space of linear perturbations. Further, the action of G on B induces an action on the space $\mathrm{Vect}(B)$ of vector fields. One may thus decompose both spaces into subspaces which correspond to irreducible representations of G. It follows from the structure of the action (2.3.3) that it leaves this decomposition invariant, i.e., perturbations which transform under a particular representation of G transform only under infinitesimal diffeomorphisms which transform under the same representation of G. Using this fact leads to a great simplification of the theory of gauge-invariant linear perturbations.

We make use of the aforementioned considerations when we study linear cosmological perturbations in section 5.1. An extension to higher order perturbations is used in section 5.2.

2.3.3 Field space symmetries

The second class of symmetry transformations we consider, and which we will call field field space symmetries, are essentially the complementary case to the base space symmetries we discussed above. While the latter are fully determined by the action of a symmetry group on the base space B of a fiber bundle $\pi : E \to B$, the former are constructed such that their action on the base space becomes trivial. In the most simple case, one may consider a group action

$$\begin{array}{rccc} \mathbf{\Psi} & : & G \times E & \to & E \\ & & (u, e) & \mapsto & \mathbf{\Psi}_u(e) \end{array} \,, \tag{2.3.4}$$

with the additional assumption that for each $u \in G$, $\mathbf{\Psi}_u : E \to E$ is a *vertical bundle automorphism*, which we defined in section 2.1.1 as a bundle automorphism covering the identity id_B, so that $\pi \circ \mathbf{\Psi}_u = \pi$. It induces an action on the space of sections as

$$\begin{array}{rccc} \tilde{\mathbf{\Psi}} & : & G \times \Gamma(\pi) & \to & \Gamma(\pi) \\ & & (u, \sigma) & \mapsto & \mathbf{\Psi}_u \circ \sigma \end{array} \,. \tag{2.3.5}$$

While this approach is sufficient for simple cases such as the conformal transformations discussed in section 6.1, it does not cover examples such as the disformal transformations studied in section 6.2, where the value of the transformed section at a given point $b \in B$ does not only depend on the value $\sigma(b)$ of the original section at the same point, but also on its derivatives. In order to define this more general class of transformations, it is more convenient to consider infinitesimal transformations. Given a one-parameter subgroup $t \mapsto \exp(-t\xi)$ of G generated by a Lie algebra element $\xi \in \mathfrak{g}$, denote by $X_\xi \in \mathrm{Vect}(E)$ the corresponding fundamental vector field, which generates the action of this one-parameter subgroup on E. Since we assumed $\mathbf{\Psi}$ to be vertical, also X_ξ must be a vertical vector field, $\pi_* \circ X_\xi = 0$. We thus have a map

$$\begin{array}{rcrcl} X & : & \mathfrak{g} \times E & \to & VE \\ & & (\xi, e) & \mapsto & X_\xi(e) \end{array} , \tag{2.3.6}$$

where $VE = \ker \pi_* \subset TE$ is the total space of the vertical tangent bundle $\nu : VE \to E$, and we have the condition that for every $\xi \in \mathfrak{g}$, X_ξ is a section of the vertical tangent bundle, hence $\nu \circ X_\xi = \mathrm{id}_E$. In order to generalize this notion such that it implements the additional dependence on the derivative of sections, one may replace the second argument by a jet, and consider maps of the form

$$\begin{array}{rcrcl} \mathbf{X} & : & \mathfrak{g} \times J^r(\pi) & \to & VE \\ & & (\xi, e) & \mapsto & X_\xi(e) \end{array} \tag{2.3.7}$$

for a given r, or to simplify the following considerations,

$$\begin{array}{rcrcl} \mathbf{X} & : & \mathfrak{g} \times J^\infty(\pi) & \to & VE \\ & & (\xi, e) & \mapsto & X_\xi(e) \end{array} , \tag{2.3.8}$$

keeping in mind that the latter may define on any finite, but unspecified number of derivatives. Also here one needs an additional condition, to guarantee that $\mathbf{X}_\xi(j_b^\infty \sigma)$ is a vector at $\sigma(b)$ for all sections $\sigma \in \Gamma(\pi)$ and $b \in B$. Hence, one demands that

$$\nu \circ \mathbf{X}_\xi = \pi_{\infty,0} , \tag{2.3.9}$$

where $\pi_{\infty,0} : J^\infty(\pi) \to E$ is the jet projection satisfying $\pi_{\infty,0}(j_b^\infty \sigma) = \sigma(b)$. A map $\mathbf{X}_\xi$ of this type is called an *evolutionary vector field* [137]. Its action on the space $\Gamma(\pi)$ of sections can most easily understood in terms of the induced group action $\tilde{\Psi} : G \times \Gamma(\pi) \to \Gamma(\pi)$. For a one-parameter group $t \mapsto \exp(-t\xi)$ one has

$$\left.\frac{\mathrm{d}}{\mathrm{d}t} \tilde{\mathbf{\Psi}}_{\exp(-t\xi)}(\sigma)(b)\right|_{t=0} = \mathbf{X}_\xi(j_b^\infty \sigma) \tag{2.3.10}$$

as the tangent vector at $t = 0$ through the curve $t \mapsto \tilde{\mathbf{\Psi}}_{\exp(-t\xi)}(\sigma)(b) \in E_b$.

2.3.4 Transformations of Lagrangians

An important difference from the earlier discussed case of base space transformation concerns the application of the construction outlined in the previous section. For the former case, we were mostly interested in sections σ which are invariant under the transformation. However, for the field space transformations we discuss here, such sections are those on which $\mathbf{X}$ acts trivially. A more interesting topic to study is the invariance of the dynamics of field theories under the action of evolutionary vector fields, or the transformation of one field theory into another under this action. Most commonly, and in particular in the case of gravity theories, the dynamics of a field theory are given by a Lagrangian. We briefly summarize this approach below.

In the following we assume that the underlying mathematical structure of the theory under consideration is given by a natural fiber bundle $\pi : E \to B$, where the base space B

is most commonly identified with the spacetime manifold M, or the total space of another fiber bundle over M, such as its tangent bundle TM. Field configurations are described by sections $\sigma : B \to E$ of this bundle, and the phase space of the theory is given as the space $\Gamma(\pi)$ of all such sections. The dynamics of the theory is defined by specifying a Lagrangian. Most formally, the latter can be defined by using the notion of jet bundles [152], either by using jet bundles of finite order [120], or infinite-dimensional jet bundles in the construction of the variational bicomplex [1]. Here we restrict the discussion to the latter framework for simplicity. In this context, a *Lagrangian* is a horizontal n-form $L \in \Omega^n(J^\infty(\pi))$ on the infinite jet bundle, where $n = \dim B$ is the dimension of the base manifold, and by horizontal it is understood that $\iota_\Xi L = 0$ for any vector field $\Xi \in \mathrm{Vect}(J^\infty(\pi))$ which satisfies $\pi_\infty \circ \Xi = 0$, where $\pi_\infty : J^\infty(\pi) \to B$ is the source projection. For any compact domain $D \subset B$ and local section $\sigma : D \to E$ the Lagrangian defines the *action functional*

$$S_D[\sigma] = \int_D (j^\infty \sigma)^* L\,. \tag{2.3.11}$$

At the heart of Lagrange theory lies the variational principle, which states that solutions to the theory defined by L are such sections σ for which the action is stationary, i.e., for which the variation of the action with respect to the section vanishes. One finds that these sections solve the *Euler-Lagrange equations*

$$(\mathcal{E}L) \circ j^\infty \sigma = 0\,, \tag{2.3.12}$$

where $\mathcal{E} : \Omega^n(J^\infty(\pi)) \to \Omega^{n+1}(J^\infty(\pi))$ is called the *Euler operator*.

We see that both evolutionary vector fields and Lagrangians are defined on the infinite jet bundle. The former, however, takes values only in the vertical tangent bundle VE. In order to obtain a vector field on $J^\infty(\pi)$, which can then be applied to a Lagrangian, one starts from the evolutionary vector field $\mathbf{X}_\xi$, where $\xi \in \mathfrak{g}$ is an element of the symmetry algebra, and defines its prolongation $\mathrm{pr}\mathbf{X}_\xi \in \mathrm{Vect}(J^\infty(\pi))$ as the unique vertical vector field on the infinite jet bundle satisfying

$$\int_D (j^\infty \sigma)^* \mathcal{L}_{\mathrm{pr}\mathbf{X}_\xi} L = \frac{\mathrm{d}}{\mathrm{d}t} \int_D \left[j^\infty \tilde{\mathbf{\Psi}}_{\exp(-t\xi)}(\sigma) \right]^* L \bigg|_{t=0} \tag{2.3.13}$$

for all sections. As in the case of base space symmetries and sections we discussed before, it is most common to study Lagrangians which are invariant under a given transformation group, which in this case means that $\mathcal{L}_{\mathrm{pr}\mathbf{X}_\xi} L$ is an exact form, and therefore does not contribute to the action integral. It follows that such transformation map solutions of the Euler-Lagrange equations (2.3.12) again to solutions. The most famous result on this topic is Noether's theorem, which provides an explicit derivation of conserved quantities from symmetries of a Lagrangian.

Another, more general possibility is to study not a single Lagrangian, but a class of Lagrangians, which is invariant under the given transformation group, such that the group elements relate different Lagrangians within the given class to each other. In this case these different Lagrangians may be regarded as generating an equivalent dynamics, but expressed in a different representation of the field variables, where the latter are related by the transformation group. We will study such classes of Lagrangians in section 6.

3 Extending the notion of symmetry

In the majority of gravity theories, most prominently general relativity, the fundamental field mediating the gravitational action is modeled as a section of a natural bundle. In this case one immediately obtains a notion for the invariance of a field configuration under the action of a transformation group on the underlying base space, as discussed in sections 2.1.2 and 2.3.1. However, also more general classes of geometries are employed, such as Cartan geometry [163, 162, 164, 47, 46, 48, 45, 58, 165, 59] or Finsler geometry [140], and also teleparallel geometry may be viewed in terms of Cartan geometry [40, 104, 103] or an additional vector bundle [4]. In order to describe symmetric gravitational fields within these theories, one therefore needs to extend the notion of symmetry to these underlying geometries. One approach would be to provide a separate definition of symmetry for any given geometry under consideration. Another, more succinct approach is to subsume the relevant geometries under a larger class, and to provide a unified notion of symmetry for this unifying class. It turns out that Cartan geometry can be used for this unification, as is shown in section 3.1. Its application to teleparallel geometry is shown in section 3.2.

3.1 Unified approach using Cartan geometry

The gravity theories which are the subject os this thesis are based on various underlying geometries, ranging from different subclasses of metric-affine geometries to Finsler geometry. In order to study these theories in a unified framework, one may therefore aim to find a unified geometric description, which encompasses the underlying geometries encountered thus far. It turns our that this can be achieved by using Cartan geometry, which has already been successfully applied to the description of various gravity theories [163, 162, 164, 47, 46, 48, 45, 58, 165, 59]. A straightforward question arising from this unification approach whether also the notion of spacetime symmetries present in the more specific symmetries enjoys a unified description in terms of Cartan geometry. This question has been studied and answered in our work [H1], which we summarize here. The general notion of symmetries for first-order reductive Cartan geometry is explained in section 3.1.1. We then discuss two special cases: metric-affine geometry in section 3.1.2 and Finsler geometry in section 3.1.3.

3.1.1 Symmetries of first-order reductive Cartan geometries

This aim to construct a general notion of symmetry for Cartan geometries is obstructed by the fact that the principal bundle, which is the most fundamental ingredient to the definition of Cartan geometry following its definition given in section 2.2.1, is in general not a natural bundle. Hence, it is not defined via a bundle functor, and there is no functorial lift of diffeomorphisms from the base manifold to the total space of the bundle, on which the Cartan connection is defined. One may therefore restrict the attention to such classes of Cartan geometries for which it is reasonable to assume that a lift may be constructed. It turns out that such as class is given by first order reductive Cartan geometries, since their principal bundle is a subbundle of the general linear frame bundle [155], for which the following construction is possible. Given a diffeomorphism $\psi : B \to B$, one can construct a lift $\hat{\psi} : \mathrm{GL}(B) \to \mathrm{GL}(B)$, which assigns to every frame $f : \mathbb{R}^n \to T_xB$ over $x \in B$ the frame

$$\hat{\psi}(f) = \psi_* \circ f \quad : \quad \begin{array}{ccc} \mathbb{R}^n & \to & T_{\psi(x)B} \\ v & \mapsto & \psi_*(f(v)) \end{array} . \tag{3.1.1}$$

One finds that $\hat{\psi}$ is a bundle isomorphism covering ψ. Using the previously introduced coordinates $(x^\mu, f_A{}^\mu)$ on the frame bundle, and writing the transformed coordinates obtained from the diffeomorphism $\hat{\psi}$ as $(x'^\mu, f'_A{}^\mu)$, where (x'^μ) are thr transformed coordinates on the base space B, this relation reads

$$f'^{\mu}_A = f_A{}^{\nu} \frac{\partial x'^{\mu}}{\partial x^{\nu}} \,. \tag{3.1.2}$$

Passing th infinitesimal transformations, it is possible to construct a lift of a vector field $X \in \mathrm{Vect}(B)$ from the base manifold to a vector field $\hat{X} \in \mathrm{Vect}(\mathrm{GL}(B))$ on the frame bundle as follows. Denoting by $\boldsymbol{\psi} : \mathbb{R} \times B \to B$ the one-parameter group of diffeomorphisms generated by X, one constructs the lift $\hat{\boldsymbol{\psi}}$ such that for every $t \in \mathbb{R}$, $\hat{\boldsymbol{\psi}}_t$ is the lift of $\boldsymbol{\psi}_t$ to $\mathrm{GL}(B)$. Then $\hat{\boldsymbol{\psi}} : \mathbb{R} \times \mathrm{GL}(B) \to \mathrm{GL}(B)$ is a one-parameter group of diffeomorphisms, whose generating vector field we denote by $\hat{X}$. Using again the coordinates $(x^{\mu}, f_A{}^{\mu})$ on the frame bundle $\mathrm{GL}(B)$, this vector field, which we call the canonical lift of X, then takes the form

$$\hat{X} = X^{\mu} \partial_{\mu} + f_A{}^{\mu} \partial_{\mu} X^{\nu} \bar{\partial}^A{}_{\nu} \tag{3.1.3}$$

in the coordinate basis (2.2.4) on the frame bundle. Using the fact that for a first order reductive Cartan geometry the principal bundle can canonically be identified with a subbundle $P \subset \mathrm{GL}(B)$ of the frame bundle, one may construct the following notion of symmetry for this class of geometries [65]:

Definition 3.1 (Finite symmetry of a first-order reductive Cartan geometry)**.** A finite symmetry of a Cartan geometry $(\pi : P \to B, \mathbf{A})$ with $P \subset \mathrm{GL}(B)$ is a diffeomorphism $\psi : B \to B$ such that $\hat{\psi}$ restricts to a diffeomorphism of P and $\hat{\psi}^* \mathbf{A} = \mathbf{A}$.

Passing to infinitesimal symmetries defined by a vector field X on B, this leads to the following notion:

Definition 3.2 (Infinitesimal symmetry of a first-order reductive Cartan geometry)**.** An infinite symmetry of a Cartan geometry $(\pi : P \to B, \mathbf{A})$ with $P \subset \mathrm{GL}(B)$ is a vector field $X \in \mathrm{Vect}(B)$ such that $\hat{X}$ is tangent to P and $\pounds_{\hat{X}} \mathbf{A} = 0$.

An interesting consequence follows from the split $\mathbf{A} = \boldsymbol{\omega} + \mathbf{e}$, where $\mathbf{e}$ is the solder form (2.2.5). Using the previously introduced coordinates for the description of the lifted diffeomorphism, we have

$$\hat{\psi}^* \mathbf{e} = f'^{-1\,A}{}_{\mu} \mathcal{Z}_A \mathrm{d}x'^{\mu} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} f^{-1\,A}{}_{\nu} \mathcal{Z}_A \frac{\partial x'^{\mu}}{\partial x^{\rho}} \mathrm{d}x^{\rho} = f^{-1\,A}{}_{\mu} \mathcal{Z}_A \mathrm{d}x^{\mu} = \mathbf{e} \tag{3.1.4}$$

and so the solder form is invariant under the complete lift of any diffeomorphism to the frame bundle. While this relation appears trivial using the coordinate expressions above, it can also straightforwardly be derived directly from the definition (3.1.1) of the lifted diffeomorphism $\hat{\psi}$. For any frame $f \in \mathrm{GL}(B)$ and tangent vector $w \in T_f \mathrm{GL}(B)$ we have

$$\begin{aligned}
(\hat{\psi}^* \mathbf{e})(w) &= \mathbf{e}(\hat{\psi}_*(w)) \\
&= [\hat{\psi}(f)^{-1} \circ \pi_* \circ \hat{\psi}_*](w) \\
&= [f^{-1} \circ \psi_*^{-1} \circ \pi_* \circ \hat{\psi}_*](w) \\
&= [f^{-1} \circ (\psi^{-1} \circ \pi \circ \hat{\psi})_*](w) \\
&= [f^{-1} \circ \pi_*](w) \\
&= \mathbf{e}(w) \,.
\end{aligned} \tag{3.1.5}$$

The steps of this derivation are explained as follows. First we use the definition of the pullback $\hat{\psi}^*$ of a differential form, which acts on vectors by first applying the pushforward

$\hat{\psi}_*$ and thereafter the original differential form. In the next step, the definition (2.2.6) is used, taking into account that $\hat{\psi}_*(w)$ is not a tangent vector at $\hat{\psi}(f)$, and so the appropriate inverse frame $\hat{\psi}(f)^{-1}$ at this point must be used. Then we apply the definition (3.1.1) of the frame bundle lift, from which follows $\hat{\psi}(f)^{-1} = f^{-1} \circ \psi_*^{-1}$. The resulting formula contains three consecutive pushforwards, which can be replaced by the joint pushforward along the concatenated maps. Finally, using the fact that $\hat{\psi}$ covers ψ, since it is a canonical lift, gives

$$\psi^{-1} \circ \pi \circ \hat{\psi} = \pi\,, \tag{3.1.6}$$

so that the diffeomorphism cancels, and one obtains the unchanged solder form.

Similarly, for any vector field $X \in \mathrm{Vect}(B)$ holds

$$\begin{aligned}
\pounds_{\hat{X}}\mathbf{e} &= \mathrm{d}(\hat{X} \lrcorner \mathbf{e}) + \hat{X} \lrcorner \mathrm{d}\mathbf{e}\\
&= \left[\mathrm{d}(f^{-1\,A}{}_{\mu}X^{\mu}) + (X^{\rho}\partial_{\rho} + f_{C}{}^{\rho}\partial_{\rho}X^{\sigma}\bar{\partial}^{C}{}_{\sigma}) \lrcorner (-f^{-1\,A}{}_{\nu}f^{-1\,B}{}_{\mu}\mathrm{d}f_{B}{}^{\nu} \wedge \mathrm{d}x^{\mu})\right]\mathcal{Z}_A\\
&= \left[f^{-1\,A}{}_{\mu}\partial_{\nu}X^{\mu}\mathrm{d}x^{\nu} - f^{-1\,A}{}_{\nu}f^{-1\,B}{}_{\mu}X^{\mu}\mathrm{d}f_{B}{}^{\nu}\right.\\
&\quad \left.+ f^{-1\,A}{}_{\nu}f^{-1\,B}{}_{\mu}X^{\mu}\mathrm{d}f_{B}{}^{\nu} - f_{B}{}^{\rho}\partial_{\rho}X^{\nu}f^{-1\,A}{}_{\nu}f^{-1\,B}{}_{\mu}\mathrm{d}x^{\mu})\right]\mathcal{Z}_A\\
&= 0\,.
\end{aligned} \tag{3.1.7}$$

Thus, we may omit the solder form in regards of symmetry, since it is sufficient to demand that the Ehresmann connection $\boldsymbol{\omega}$ is invariant under the (finite or infinitesimal) symmetry transformation. We will use this finding in the following sections.

3.1.2 Relation with metric-affine geometry

As a simple example, we apply the notion of symmetry for a first-order reductive Cartan geometry outlined in the previous section to the metric affine spacetime geometry introduced in section 2.2.2. In this case, the base manifold of the Cartan geometry is identified with the spacetime manifold $B = M$, while the bundle on which the Cartan geometry is defined, is the whole frame bundle $P = \mathrm{GL}(M)$. Hence, the frame bundle lift $\hat{\psi} : \mathrm{GL}(M) \to \mathrm{GL}(M)$ of any diffeomorphism $\psi : M \to M$ trivially preserves P. Similarly, the frame bundle lift $\hat{X} \in \mathrm{Vect}(\mathrm{GL}(M))$ of any vector field $X \in \mathrm{Vect}(M)$ is trivially tangent to P.

It is illustrative to derive explicitly the symmetry conditions for the Cartan connection (2.2.15). Following the finding from the previous section, that the solder form $\mathbf{e}$ is invariant under any finite or infinite transformation on the base space, it is sufficient to consider only the Ehresmann connection (2.2.16). Its pullback along the frame bundle lift $\hat{\psi}$ reads

$$\begin{aligned}
\hat{\psi}^*\boldsymbol{\omega} &= f'^{-1\,A}{}_{\mu}(\mathrm{d}f'_{B}{}^{\mu} + f'_{B}{}^{\nu}\Gamma^{\mu}{}_{\nu\rho}(x')\mathrm{d}x'^{\rho})\mathcal{H}_A{}^B\\
&= \frac{\partial x^{\mu}}{\partial x'^{\tau}}f^{-1\,A}{}_{\mu}\left(\frac{\partial x'^{\tau}}{\partial x^{\lambda}}\mathrm{d}f_{B}{}^{\lambda} + f_{B}{}^{\lambda}\frac{\partial^2 x'^{\tau}}{\partial x^{\kappa}\partial x^{\lambda}}\mathrm{d}x^{\kappa} + \frac{\partial x'^{\omega}}{\partial x^{\nu}}\frac{\partial x'^{\sigma}}{\partial x^{\rho}}f_{B}{}^{\nu}\Gamma^{\tau}{}_{\omega\sigma}(x')\mathrm{d}x^{\rho}\right)\mathcal{H}_A{}^B\\
&= f^{-1\,A}{}_{\mu}\left[\mathrm{d}f_{B}{}^{\mu} + f_{B}{}^{\nu}\frac{\partial x^{\mu}}{\partial x'^{\tau}}\left(\frac{\partial^2 x'^{\tau}}{\partial x^{\nu}\partial x^{\rho}} + \frac{\partial x'^{\omega}}{\partial x^{\nu}}\frac{\partial x'^{\sigma}}{\partial x^{\rho}}\Gamma^{\tau}{}_{\omega\sigma}(x')\right)\mathrm{d}x^{\rho}\right]\mathcal{H}_A{}^B\\
&= f^{-1\,A}{}_{\mu}\left[\mathrm{d}f_{B}{}^{\mu} + f_{B}{}^{\nu}(\psi^*\Gamma)^{\tau}{}_{\omega\sigma}(x)\mathrm{d}x^{\rho}\right]\mathcal{H}_A{}^B\,,
\end{aligned} \tag{3.1.8}$$

and so we find simply the usual transformation law

$$(\psi^*\Gamma)^{\mu}{}_{\nu\rho}(x) = \Gamma^{\tau}{}_{\omega\sigma}(x')\frac{\partial x^{\mu}}{\partial x'^{\tau}}\frac{\partial x'^{\omega}}{\partial x^{\nu}}\frac{\partial x'^{\sigma}}{\partial x^{\rho}} + \frac{\partial x^{\mu}}{\partial x'^{\tau}}\frac{\partial^2 x'^{\tau}}{\partial x^{\nu}\partial x^{\rho}} \tag{3.1.9}$$

for the coefficients of an affine connection under the action of the diffeomorphism ψ. This relationship can also be derived for the infinitesimal transformation induced by a vector field $X \in \mathrm{Vect}(M)$. In this case calculating the Lie derivative $\pounds_{\hat{X}}\boldsymbol{\omega}$, which we omit here for brevity, reveals the Lie derivative of the affine connection [166]

$$\begin{aligned}(\pounds_X\Gamma)^{\mu}{}_{\nu\rho} &= X^{\sigma}\partial_{\sigma}\Gamma^{\mu}{}_{\nu\rho} - \partial_{\sigma}X^{\mu}\Gamma^{\sigma}{}_{\nu\rho} + \partial_{\nu}X^{\sigma}\Gamma^{\mu}{}_{\sigma\rho} + \partial_{\rho}X^{\sigma}\Gamma^{\mu}{}_{\nu\sigma} + \partial_{\nu}\partial_{\rho}X^{\mu} \\ &= \nabla_{\rho}\nabla_{\nu}X^{\mu} - X^{\sigma}R^{\mu}{}_{\nu\rho\sigma} - \nabla_{\rho}(X^{\sigma}T^{\mu}{}_{\nu\sigma})\,,\end{aligned} \tag{3.1.10}$$

which is simply the infinitesimal analogue of the previously derived and well-known transformation formula. The second line shown here, which is straightforward to obtain from the previous one, shows that it is a tensor, since the connection bundle is an affine bundle modeled over a tensor bundle, and the Lie derivative takes its values in the latter.

In addition to the affine connection, which defines the Cartan connection, a metric-affine geometry is also equipped with a metric, which we lifted to a scalar quantity (2.2.18) on the frame bundle. Hence, one also demands this quantity to be invariant under the frame bundle lift of a symmetry transformation, in order for the full metric-affine geometry to be regarded as invariant. For the finite transformation, one thus calculates the pullback

$$\hat{\psi}^*\mathbf{g} = f'_A{}^{\mu}f'_B{}^{\nu}g_{\mu\nu}(x')\mathcal{Z}^A \otimes \mathcal{Z}^B = \frac{\partial x'^{\rho}}{\partial x^{\mu}}\frac{\partial x'^{\sigma}}{\partial x^{\nu}}f_A{}^{\mu}f_B{}^{\nu}g_{\rho\sigma}(x')\mathcal{Z}^A \otimes \mathcal{Z}^B\,, \tag{3.1.11}$$

which resembles the pullback

$$(\psi^*g)_{\mu\nu}(x) = g_{\rho\sigma}(x')\frac{\partial x'^{\rho}}{\partial x^{\mu}}\frac{\partial x'^{\sigma}}{\partial x^{\nu}} \tag{3.1.12}$$

of the metric tensor. Similarly, the Lie derivative yields

$$\begin{aligned}\pounds_{\hat{X}}\mathbf{g} &= \hat{X} \lrcorner \mathrm{d}\mathbf{g} \\ &= \left(X^{\lambda}\partial_{\lambda} + f_D{}^{\sigma}\partial_{\sigma}X^{\lambda}\bar{\partial}^D{}_{\lambda}\right) \\ &\quad \lrcorner \left[(g_{\rho\nu}f^{-1}{}^C{}_{\mu} + g_{\mu\rho}f^{-1}{}^C{}_{\nu})\mathrm{d}f_C{}^{\rho} + \partial_{\rho}g_{\mu\nu}\mathrm{d}x^{\rho}\right] f_A{}^{\mu}f_B{}^{\nu}\mathcal{Z}^A \otimes \mathcal{Z}^B \\ &= \left[f_C{}^{\sigma}\partial_{\sigma}X^{\rho}(g_{\rho\nu}f^{-1}{}^C{}_{\mu} + g_{\mu\rho}f^{-1}{}^C{}_{\nu}) + X^{\rho}\partial_{\rho}g_{\mu\nu}\right] f_A{}^{\mu}f_B{}^{\nu}\mathcal{Z}^A \otimes \mathcal{Z}^B \\ &= \left(\partial_{\mu}X^{\rho}g_{\rho\nu} + \partial_{\nu}X^{\rho}g_{\mu\rho} + X^{\rho}\partial_{\rho}g_{\mu\nu}\right) f_A{}^{\mu}f_B{}^{\nu}\mathcal{Z}^A \otimes \mathcal{Z}^B\,,\end{aligned} \tag{3.1.13}$$

from which we see the Lie derivative

$$(\pounds_X g)_{\mu\nu} = X^{\rho}\partial_{\rho}g_{\mu\nu} + \partial_{\mu}X^{\rho}g_{\rho\nu} + \partial_{\nu}X^{\rho}g_{\mu\rho}\,, \tag{3.1.14}$$

as one may expect. Hence, we find that the invariance of a metric-affine geometry under the action of a transformation group which we derived from the unified Cartan geometry approach simply agrees with the conventional notion of invariance in terms of the metric and affine connection. It is then straightforward to specialize this notion of symmetry to particular subclasses of metric-affine geometries; this is done for the case of teleparallel geometry in section 3.2.

3.1.3 Relation with Finsler geometry

As the second example we consider the application of our notion of symmetry for first-order reductive Cartan geometries to Finsler geometry, following its definition in section 2.2.4. The main difficulty in this case arises from the fact that the characteristic objects describing the properties of a Finsler geometry are not sections of natural bundles over the spacetime manifold M, on which the spacetime symmetry group acts, but natural bundles over its tangent bundle TM. Hence, an additional, intermediate step is required, in order to lift a diffeomorphism $\psi : M \to M$ to $\hat{\psi} = \psi_* : TM \to TM$. In the following one then only considers the transformation of Finsler geometric objects under these induced diffeomorphisms $\hat{\psi}$ of the tangent bundle, instead of the full diffeomorphism group of the total space manifold TM. From this restriction follows that the transformation of these objects takes a particularly simple form, which we briefly show below. The starting point for deriving these transformations is the transformation of the geometry function L. Writing the coordinates of a point in TM as $(x^\mu, \bar{x}^\mu)$ and its image under $\hat{\psi}$ as $(x'^\mu, \bar{x}'^\mu)$, one has the relation

$$\bar{x}'^\mu = \bar{x}^\nu \frac{\partial x'^\mu}{\partial x^\nu} . \tag{3.1.15}$$

The pullback of L along $\hat{\psi}$ then simply reads

$$(\hat{\psi}^* L)(x, \bar{x}) = L(x', \bar{x}') . \tag{3.1.16}$$

The transformation of the remaining quantities is then derived from this relation. For the Hessian (2.2.38) and the Finsler metric (2.2.40) one finds the common transformation behavior

$$(\hat{\psi}^* g^{L/F})_{\mu\nu}(x, \bar{x}) = g^{L/F}_{\rho\sigma}(x', \bar{x}') \frac{\partial x'^\rho}{\partial x^\mu} \frac{\partial x'^\sigma}{\partial x^\nu} , \tag{3.1.17}$$

which shows that they transform analogously to tensor fields on the base manifold, which is a characteristic property of d-tensor fields. This is to be contrasted with the transformation of the coefficients (2.2.42) of the non-linear connection, which reads

$$(\hat{\psi}^* N)^\mu{}_\nu(x, \bar{x}) = N^\rho{}_\sigma(x', \bar{x}') \frac{\partial x^\mu}{\partial x'^\rho} \frac{\partial x'^\sigma}{\partial x^\nu} + \bar{x}^\sigma \frac{\partial^2 x'^\rho}{\partial x^\nu \partial x^\sigma} \frac{\partial x^\mu}{\partial x'^\rho} , \tag{3.1.18}$$

and which is inhomogeneous, as can be seen from the appearance of an additional term. Finally, for the coefficients (2.2.46) one finds the transformation rules

$$(\hat{\psi}^* F)^\mu{}_{\nu\rho}(x, \bar{x}) = F^\sigma{}_{\tau\omega}(x', \bar{x}') \frac{\partial x^\mu}{\partial x'^\sigma} \frac{\partial x'^\tau}{\partial x^\nu} \frac{\partial x'^\omega}{\partial x^\rho} + \frac{\partial x^\nu}{\partial x'^\sigma} \frac{\partial^2 x'^\sigma}{\partial x^\nu \partial x^\rho} , \tag{3.1.19a}$$

$$(\hat{\psi}^* C)^\mu{}_{\nu\rho}(x, \bar{x}) = C^\sigma{}_{\tau\omega}(x', \bar{x}') \frac{\partial x^\mu}{\partial x'^\sigma} \frac{\partial x'^\tau}{\partial x^\nu} \frac{\partial x'^\omega}{\partial x^\rho} , \tag{3.1.19b}$$

so that only the latter constitutes the components of a d-tensor field, while the former receives an additional inhomogeneous contribution.

In order to relate this set of transformation rules to the description in terms of Cartan geometry, it is more convenient to consider infinitesimal diffeomorphisms generated by a vector field $X \in \mathrm{Vect}(M)$ instead. Using the fact that TM is a natural bundle, one can construct the complete lift $\hat{X} \in \mathrm{Vect}(TM)$ given by

$$\hat{X} = X^\mu \partial_\mu + \bar{x}^\nu \partial_\nu X^\mu \bar{\partial}_\mu . \tag{3.1.20}$$

For the Finsler geometric objects defined on TM, it is now straightforward to calculate their Lie derivatives, i.e., their change under an infinitesimal transformation generated by $\hat{X}$. For the scalar function L, this takes the obvious form

$$\mathcal{L}_{\hat{X}} L = X^\mu \partial_\mu L + \bar{x}^\nu \partial_\nu X^\mu \bar{\partial}_\mu L , \tag{3.1.21}$$

which is simply the directional derivative along $\hat{X}$. For the Hessian (2.2.38) and Finsler metric (2.2.40), which are d-tensors, one finds the formula

$$(\pounds_{\hat{X}} g^{L/F})_{\mu\nu} = X^\rho \partial_\rho g^{L/F}_{\mu\nu} + \bar{x}^\sigma \partial_\sigma X^\rho \bar{\partial}_\rho g^{L/F}_{\mu\nu} + \partial_\mu X^\rho g^{L/F}_{\rho\nu} + \partial_\nu X^\rho g^{L/F}_{\mu\rho}\,, \tag{3.1.22}$$

which is reminiscent of the usual Lie derivative of tensor fields defined on the base manifold. This is different for the coefficients (2.2.42) of the non-linear connection, which transform as

$$(\pounds_{\hat{X}} N)^\mu{}_\nu = X^\rho \partial_\rho N^\mu{}_\nu + \bar{x}^\sigma \partial_\sigma X^\rho \bar{\partial}_\rho N^\mu{}_\nu - \partial_\rho X^\mu N^\rho{}_\nu + \partial_\nu X^\rho N^\mu{}_\rho + \bar{x}^\rho \partial_\nu \partial_\rho X^\mu\,, \tag{3.1.23}$$

so that one obtains an inhomogeneous contribution, as one should expect. This can also be seen for the coefficients (2.2.46) of the Cartan linear connection, which reads

$$\begin{aligned}(\pounds_{\hat{X}} F)^\mu{}_{\nu\rho} &= X^\sigma \partial_\sigma F^\mu{}_{\nu\rho} + \bar{x}^\tau \partial_\tau X^\sigma \bar{\partial}_\sigma F^\mu{}_{\nu\rho} - \partial_\sigma X^\mu F^\sigma{}_{\nu\rho} + \partial_\nu X^\sigma F^\mu{}_{\sigma\rho} + \partial_\rho X^\sigma F^\mu{}_{\nu\sigma}\\ &\quad + \partial_\nu \partial_\rho X^\mu\,, \qquad (3.1.24a)\\ (\pounds_{\hat{X}} C)^\mu{}_{\nu\rho} &= X^\sigma \partial_\sigma C^\mu{}_{\nu\rho} + \bar{x}^\tau \partial_\tau X^\sigma \bar{\partial}_\sigma C^\mu{}_{\nu\rho} - \partial_\sigma X^\mu C^\sigma{}_{\nu\rho} + \partial_\nu X^\sigma C^\mu{}_{\sigma\rho} + \partial_\rho X^\sigma C^\mu{}_{\nu\sigma}\,, \qquad (3.1.24b)\end{aligned}$$

and thus confirms that only $C^\mu{}_{\nu\rho}$ transforms as a d-tensor, while $F^\mu{}_{\nu\rho}$ does not.

Of particular interest are such diffeomorphisms $\psi : M \to M$, which leave the geometry function L invariant, $\hat{\psi}^* L = L$. From the fact that all other geometric objects we discussed above are derived from the geometry function that also these objects are invariant under the induced action of ψ if L is invariant. This justifies to regard ψ as a (finite) symmetry of a Finsler spacetime if and only if $\hat{\psi}^* L = L$. The analogous statement holds also for infinitesimal transformations generated by a vector field $X \in \mathrm{Vect}(M)$: if the Lie derivative $\pounds_{\hat{X}} L$ of the Finsler geometry function L vanishes, then also the Lie derivatives of the remaining geometric objects mentioned above vanish. Hence, we regard X as an infinitesimal symmetry if and only if $\pounds_{\hat{X}} = 0$. In the following, we will restrict ourselves to the case of infinitesimal symmetries.

As discussed in section 2.2.4, Finsler spacetimes give rise to a first-order reductive Cartan geometry, whose Cartan connection (2.2.52) is defined on the total space of a principal K-bundle $\pi : P \to O$, where $K = \mathrm{SO}(3)$ and O is the observer space, and P can be identified with a subbundle of the frame bundle $\mathrm{GL}(O)$. Relating the symmetries of this observer space Cartan geometry to those of the underlying Finsler geometry poses a number of conceptual difficulties. The main difficulty arises from the fact that the former is generated by a vector field $\Xi \in \mathrm{Vect}(O)$ on the observer space, while the latter is generated by a vector field $X \in \mathrm{Vect}(M)$, and there is no direct relation between $\mathrm{Vect}(M)$ and $\mathrm{Vect}(O)$. However, recall that $O \subset TM$ is defined as a connected component of the level set of L where $L = 1$. If X is a symmetry of the Finsler spacetime, so that its complete lift $\hat{X} \in \mathrm{Vect}(TM)$ to the tangent bundle satisfies $\pounds_{\hat{X}} L = 0$, it thus follows that $\hat{X}$ is tangent to O, and thus restricts to a vector field $\Xi = \hat{X}|_O$. A lengthy, but straightforward calculation shows that the canonical lift of this vector field Ξ to $\mathrm{GL}(O)$ is tangent to P and preserves the Cartan connection, so that Ξ is a symmetry of the observer space Cartan geometry.

The fact that a symmetry $X \in \mathrm{Vect}(M)$ of a Finsler spacetime induces a symmetry $\Xi \in \mathrm{Vect}(O)$ of the corresponding observer space Cartan geometry does not come as a surprise, since the observer space and the Cartan connection are fully determined by the geometry function L defining the Finsler spacetime. More remarkable is the fact that also the converse statement holds: every symmetry $\Xi \in \mathrm{Vect}(O)$ of the observer space Cartan geometry induced from a Finsler spacetime originates from a symmetry $X \in \mathrm{Vect}(M)$ of the Finsler spacetime itself. Together with the former statement it thus follows that there exists a one-to-one correspondence between symmetries $X \in \mathrm{Vect}(M)$ of Finsler spacetimes and $\Xi \in \mathrm{Vect}(O)$ of their induced observer space Cartan geometry. The proof of this statement is rather lengthy and shown in our work [H1].

3.2 Symmetries in teleparallel geometry

A particular subclass of the general class of orthogonal Cartan geometries discussed in the previous section is the teleparallel geometry, which can be described by a metric $g_{\mu\nu}$, together with a flat, metric-compatible affine connection $\overset{\bullet}{\Gamma}{}^{\mu}{}_{\nu\rho}$ [11]. More commonly, however, a teleparallel geometry is described in the covariant formulation [121] by using a tetrad $\theta^A{}_\mu$, together with a flat, metric-compatible spin connection $\overset{\bullet}{\omega}{}^A{}_{B\mu}$, from which the metric and affine connection are obtained via the definitions (2.2.22) and (2.2.27). Hence, the question arises how the notion of symmetry derived in our work [H1] can be expressed in terms of these more common field variables. We answered this question in detail in our work [H2], and determined the teleparallel geometries which are invariant under a number of different symmetry groups. We summarize the general construction and derivation of symmetry conditions below. In section 3.2.1, we consider finite group actions, before we pass to infinitesimal actions in section 3.2.2. In both sections we work in a general Lorentz gauge. Finally, in section 3.2.3, we translate the results into the Weitzenböck gauge, which simplifies the conditions significantly, and show their equivalence with the general case.

3.2.1 Finite symmetries

To derive the symmetry condition for the tetrad, one starts with the symmetry condition (3.1.12) for the metric. It follows from the definition (2.2.22) that the metric $g_{\mu\nu}$ obtained from a tetrad $\theta^A{}_\mu$ is invariant under a diffeomorphism $\psi : M \to M$ if and only if the tetrad and its pullback $(\psi^*\theta)^A{}_\mu$ are related by a local Lorentz transformation $\Lambda : M \to \mathrm{SO}(1,3)$. Hence, we can define the symmetry condition for the tetrad as

$$(\psi^*\theta)^A{}_\mu = \Lambda^A{}_B\theta^B{}_\mu\,. \tag{3.2.1}$$

It then follows further that the affine connection (2.2.27) satisfies the symmetry condition (3.1.9) if and only if the pullback of the spin connection is given by

$$(\psi^*\overset{\bullet}{\omega})^A{}_{B\mu} = \Lambda^A{}_C\left[(\Lambda^{-1})^D{}_B\overset{\bullet}{\omega}{}^C{}_{D\mu} + \partial_\mu(\Lambda^{-1})^C{}_B\right]\,. \tag{3.2.2}$$

Instead of a single diffeomorphism $\psi : M \to M$, it is more common to consider invariance under the action $\boldsymbol{\psi} : G \times M \to M$ of a group G by diffeomorphisms on M. In this case, the symmetry conditions (3.2.1) and (3.2.2), with $\psi = \boldsymbol{\psi}_u$, must be satisfied for all $u \in G$. Here Λ may be chosen differently for different $u \in G$, and so the single local Lorentz transformation Λ is replaced by a map $\boldsymbol{\Lambda} : G \times M \to \mathrm{SO}(1,3)$. However, it turns out that this map is not arbitrary. This follows from the fact that $\boldsymbol{\psi}$ is a (left) group action, and hence by definition satisfies $\boldsymbol{\psi}_{uv} = \boldsymbol{\psi}_u \circ \boldsymbol{\psi}_v$ for all $u, v \in G$. For $\boldsymbol{\Lambda}_{uv}$ thus follows

$$\begin{aligned}
\boldsymbol{\Lambda}^A_{uv\,B}\theta^B{}_\mu &= (\boldsymbol{\psi}^*_{uv}\theta)^A{}_\mu \\
&= (\boldsymbol{\psi}^*_v\boldsymbol{\psi}^*_u\theta)^A{}_\mu \\
&= [\boldsymbol{\psi}^*_v(\boldsymbol{\Lambda}_u \cdot \theta)]^A{}_\mu \\
&= (\boldsymbol{\psi}^*_v\boldsymbol{\Lambda}_u)^A{}_B(\boldsymbol{\psi}^*_v\theta)^B{}_\mu \\
&= (\boldsymbol{\Lambda}_u \circ \boldsymbol{\psi}_v)^A{}_B\boldsymbol{\Lambda}^B_{v\,C}\theta^C{}_\mu\,,
\end{aligned} \tag{3.2.3}$$

and so we find

$$\mathbf{\Lambda}^A_{uv\,B} = (\mathbf{\Lambda}_u \circ \boldsymbol{\psi}_v)^A{}_C \mathbf{\Lambda}^C_{v\,B} \,. \tag{3.2.4}$$

We call this map $\mathbf{\Lambda}$, which intertwines the group operation of the Lorentz group with the action of the symmetry group a *local Lie group homomorphism*.

3.2.2 Infinitesimal symmetries

By passing from the general group action discussed above to one-parameter groups, one easily obtains the corresponding infinitesimal symmetry conditions. One finds that the metric satisfies the condition (3.1.14) of invariance under the one-parameter diffeomorphism group generated by a vector field X^μ if and only if the tetrad satisfies

$$(\mathcal{L}_X \theta)^A{}_\mu = -\lambda^A{}_B \theta^B{}_\mu \,, \tag{3.2.5}$$

where $\lambda : M \to \mathfrak{so}(1,3)$ is an infinitesimal local Lorentz transformation. Similarly, for the spin connection, the condition (3.1.10) yields

$$(\mathcal{L}_X \overset{\bullet}{\omega})^A{}_{B\mu} = \partial_\mu \lambda^A{}_B + \overset{\bullet}{\omega}{}^A{}_{C\mu} \lambda^C{}_B - \overset{\bullet}{\omega}{}^C{}_{B\mu} \lambda^A{}_C = \overset{\bullet}{\mathrm{D}} \lambda^A{}_B \,, \tag{3.2.6}$$

in terms of the exterior covariant derivative $\overset{\bullet}{\mathrm{D}}$. In order to obtain the full action of the Lie algebra $\mathfrak{g}$ of the symmetry group G, let $\mathbf{X} : \mathfrak{g} \to \mathrm{Vect}(M)$ denote the fundamental vector fields. For two algebra elements $\xi, \zeta \in \mathfrak{g}$ we have

$$\begin{aligned}
\boldsymbol{\lambda}^A_{[\xi,\zeta]\,B} \theta^B{}_\mu &= -(\mathcal{L}_{\mathbf{X}_{[\xi,\zeta]}} \theta)^A{}_\mu \\
&= -(\mathcal{L}_{[\mathbf{X}_\xi, \mathbf{X}_\zeta]} \theta)^A{}_\mu \\
&= -(\mathcal{L}_{\mathbf{X}_\xi} \mathcal{L}_{\mathbf{X}_\zeta} \theta)^A{}_\mu + (\mathcal{L}_{\mathbf{X}_\zeta} \mathcal{L}_{\mathbf{X}_\xi} \theta)^A{}_\mu \\
&= [\mathcal{L}_{\mathbf{X}_\xi} (\boldsymbol{\lambda}_\zeta \cdot \theta)]^A{}_\mu - [\mathcal{L}_{\mathbf{X}_\zeta} (\boldsymbol{\lambda}_\xi \cdot \theta)]^A{}_\mu \\
&= \left[(\mathbf{X}_\xi \boldsymbol{\lambda}_\zeta)^A{}_B \theta^B{}_\mu - (\mathbf{X}_\zeta \boldsymbol{\lambda}_\xi)^A{}_B \theta^B{}_\mu + \boldsymbol{\lambda}^A_{\zeta\,B} (\mathcal{L}_{\mathbf{X}_\xi} \theta)^B{}_\mu - \boldsymbol{\lambda}^A_{\xi\,B} (\mathcal{L}_{\mathbf{X}_\zeta} \theta)^B{}_\mu \right] \\
&= \left[(\mathbf{X}_\xi \boldsymbol{\lambda}_\zeta)^A{}_B \theta^B{}_\mu - (\mathbf{X}_\zeta \boldsymbol{\lambda}_\xi)^A{}_B \theta^B{}_\mu - \boldsymbol{\lambda}^A_{\zeta\,B} \boldsymbol{\lambda}^B_{\xi\,C} \theta^C{}_\mu + \boldsymbol{\lambda}^A_{\xi\,B} \boldsymbol{\lambda}^B_{\zeta\,C} \theta^C{}_\mu \right] \,,
\end{aligned} \tag{3.2.7}$$

from which follows

$$\boldsymbol{\lambda}^A_{[\xi,\zeta]\,B} = \left([\boldsymbol{\lambda}_\xi, \boldsymbol{\lambda}_\zeta] + \mathbf{X}_\xi \boldsymbol{\lambda}_\zeta - \mathbf{X}_\zeta \boldsymbol{\lambda}_\xi \right)^A{}_B \,. \tag{3.2.8}$$

In analogy to the terminology introduced in the finite transformation case, we call a map $\boldsymbol{\lambda}$ which satisfies this relation a *local Lie algebra homomorphism*.

3.2.3 Local Lorentz transformation and Weitzenböck gauge

The rather cumbersome dependence of the local homomorphisms on the manifold M, which enters through the pullback along $\boldsymbol{\psi}$ in the finite case (3.2.4), and the derivative along the fundamental vector fields $\mathbf{X}$ in the infinitesimal case (3.2.8), can be simplified by performing a local Lorentz transformation (2.2.28) to the Weitzenböck gauge, so that the transformed spin connection vanishes, $\overset{\bullet}{\omega}{}'^A{}_{B\mu} = 0$. In this gauge the invariance condition (3.2.2) under the finite diffeomorphism $\psi = \boldsymbol{\psi}_u$ reads

$$0 = (\boldsymbol{\psi}^*_u \overset{\bullet}{\omega}{}')^A{}_{B\mu} = \mathbf{\Lambda}'^A_{u\,C} \left[(\mathbf{\Lambda}'^{-1}_u)^D{}_B \overset{\bullet}{\omega}{}'^C{}_{D\mu} + \partial_\mu (\mathbf{\Lambda}'^{-1}_u)^C{}_B \right] = \mathbf{\Lambda}'^A_{u\,C} \partial_\mu (\mathbf{\Lambda}'^{-1}_u)^C{}_B \,, \tag{3.2.9}$$

and so $\partial_\mu \mathbf{\Lambda}'^A_{u\,B} = 0$ for all $u \in G$. Similarly, the infinitesimal invariance condition (3.2.6) with $\xi = \mathbf{X}_\xi$ transforms to

$$0 = (\mathcal{L}_{\mathbf{X}_\xi} \overset{\bullet}{\omega}{}')^A{}_{B\mu} = \partial_\mu \boldsymbol{\lambda}'^A_{\xi\,B} + \overset{\bullet}{\omega}{}'^A{}_{C\mu} \boldsymbol{\lambda}'^C_{\xi\,B} - \overset{\bullet}{\omega}{}'^C{}_{B\mu} \boldsymbol{\lambda}'^A_{\xi\,C} = \partial_\mu \boldsymbol{\lambda}'^A_{\xi\,B} \,. \tag{3.2.10}$$

Hence, in the Weitzenböck gauge $\boldsymbol{\Lambda}'_u \circ \boldsymbol{\psi}_v = \boldsymbol{\Lambda}'_u$ and $\mathbf{X}_\xi \boldsymbol{\lambda}'_\zeta = 0$, so that the relations (3.2.4) and (3.2.8) simply read

$$\boldsymbol{\Lambda}'^A_{uv}{}_B = \boldsymbol{\Lambda}'^A_u{}_C \boldsymbol{\Lambda}'^C_v{}_B\,, \quad \boldsymbol{\lambda}'^A_{[\xi,\zeta]}{}_B = [\boldsymbol{\lambda}'_\xi, \boldsymbol{\lambda}'_\zeta]^A{}_B\,, \tag{3.2.11}$$

and thus are the conditions for $\boldsymbol{\Lambda}' : G \to \mathrm{SO}(1,3)$ and $\boldsymbol{\lambda}' : \mathfrak{g} \to \mathfrak{so}(1,3)$ to be homomorphisms of Lie groups and Lie algebras, respectively. Finally, the relation to the original, local homomorphisms is easily derived as

$$\begin{aligned}
\boldsymbol{\Lambda}^A_u{}_B \theta^B{}_\mu &= (\boldsymbol{\psi}^*_u \theta)^A{}_\mu \\
&= [\boldsymbol{\psi}^*_u (\Lambda^{-1} \cdot \theta')]^A{}_\mu \\
&= (\Lambda^{-1} \circ \boldsymbol{\psi}_u)^A{}_B (\boldsymbol{\psi}^*_u \theta')^B{}_\mu \\
&= (\Lambda^{-1} \circ \boldsymbol{\psi}_u)^A{}_B \boldsymbol{\Lambda}'^B_u{}_C \theta'^C{}_\mu \\
&= (\Lambda^{-1} \circ \boldsymbol{\psi}_u)^A{}_B \boldsymbol{\Lambda}'^B_u{}_C \Lambda^C{}_D \theta^D{}_\mu
\end{aligned} \tag{3.2.12}$$

and

$$\begin{aligned}
\boldsymbol{\lambda}^A_\xi{}_B \theta^B{}_\mu &= -(\pounds_{\mathbf{X}_\xi} \theta)^A{}_\mu \\
&= -[\pounds_{\mathbf{X}_\xi} (\Lambda^{-1} \cdot \theta')]^A{}_\mu \\
&= (\Lambda^{-1})^A{}_B \left[(\mathbf{X}_\xi \Lambda^B{}_C)(\Lambda^{-1})^C{}_D \theta'^D{}_\mu - (\pounds_{\mathbf{X}_\xi} \theta')^B{}_\mu\right] \\
&= (\Lambda^{-1})^A{}_B \left[\boldsymbol{\lambda}'^B_\xi{}_C + (\mathbf{X}_\xi \Lambda^B{}_D)(\Lambda^{-1})^D{}_C\right] \theta'^C{}_\mu \\
&= (\Lambda^{-1})^A{}_B \left(\boldsymbol{\lambda}'^B_\xi{}_D \Lambda^D{}_C + \mathbf{X}_\xi \Lambda^B{}_C\right) \theta^C{}_\mu\,,
\end{aligned} \tag{3.2.13}$$

from which one reads off

$$\boldsymbol{\Lambda}^A_u{}_B = (\Lambda^{-1} \circ \boldsymbol{\psi}_u)^A{}_C \boldsymbol{\Lambda}'^C_u{}_D \Lambda^D{}_B\,, \tag{3.2.14a}$$
$$\boldsymbol{\lambda}^A_\xi{}_B = (\Lambda^{-1})^A{}_C \left(\boldsymbol{\lambda}'^C_\xi{}_D \Lambda^D{}_B + \mathbf{X}_\xi \Lambda^C{}_B\right)\,. \tag{3.2.14b}$$

Note in particular that the latter broadly resembles the local Lorentz transformation (2.2.28) of the spin connection, which also takes values in the Lorentz algebra. A tedious, but straightforward calculation shows that these satisfy the original conditions (3.2.4) and (3.2.8) for the local homomorphisms. This shows that one can find the general form of any teleparallel geometry which is invariant under a given group action by first solving the symmetry conditions in the Weitzenböck gauge, where $\boldsymbol{\Lambda}'$ and $\boldsymbol{\lambda}'$ are global homomorphisms, and then perform an arbitrary local Lorentz transformation to transform the result into any other Lorentz gauge. This method is employed for cosmological symmetry in section 4.2. Further examples are given in our work [H2].

4 Exact spacetime symmetries

In the previous sections we have derived conditions of invariance under the action of a symmetry group for various geometries, which are used to model spacetime in gravity theory. We now present the application of these conditions to a selection of geometries and symmetry groups. In section 4.1 we present our work [H3], in which the most general metric-affine geometry with spherical symmetry is studied, along with various special cases. In section 4.2, the most general class of teleparallel geometries with cosmological symmetry is shown, which was derived in our work [H4].

4.1 Spherical symmetry in metric-affine geometry

In section 2.2.2, we gave an overview of metric-affine geometry and its characterizing, tensorial properties. Since the fundamental fields defining this geometry are a metric and an affine connection, which can be described in terms of first-order reductive Cartan geometry, the notion of symmetry studied in section 3.1 can directly be applied. This has been done in our work [H3], where we derived the most general class of metric-affine geometries with spherical symmetry, and a number of subclasses, which are obtained by demanding that one or several of its tensor properties (curvature, torsion, nonmetricity) vanish. Here we summarize this work as follows. In section 4.1.1, we briefly discuss our notion of spherical symmetry, including symmetry both under rotations and reflections. The most general metric-affine geometry satisfying this symmetry is shown in section 4.1.2. In section 4.1.3, we display the characterizing tensors for this geometry. Finally, we discuss special cases in section 4.1.4, which are characterized by the vanishing of one or more of these tensorial quantities.

4.1.1 Rotational and reflectional symmetries

The notion of spherical symmetry as invariance under the action of the orthogonal group O(3) may be described in terms of two disjoint symmetries. First, one may consider the symmetry group SO(3) of pure rotations. This is a connected, compact Lie group, which is generated by its Lie algebra $\mathfrak{so}(3)$. Its action on a spacetime manifold M can therefore be expressed in terms of the fundamental vector fields, which generate rotations. Using spherical coordinates $(t, r, \vartheta, \varphi)$, these are given by the three vector fields

$$R_1 = \sin\varphi\partial_\vartheta + \frac{\cos\varphi}{\tan\vartheta}\partial_\varphi\,, \tag{4.1.1a}$$

$$R_2 = -\cos\varphi\partial_\vartheta + \frac{\sin\varphi}{\tan\vartheta}\partial_\varphi\,, \tag{4.1.1b}$$

$$R_3 = -\partial_\varphi\,. \tag{4.1.1c}$$

The full group O(3), however, consists of two connected components, and so an infinitesimal description of its action is not sufficient. In addition, one needs to specify also the action of at least one group element which does not belong to the connected component of the identity. The most straightforward element to consider is the point reflection, whose action in spherical coordinates can be expressed as

$$(t, r, \vartheta, \varphi) \mapsto (t', r', \vartheta', \varphi') = (t, r, \pi - \vartheta, \varphi + \pi)\,, \tag{4.1.2}$$

where the translation of the azimuth angle φ is understood modulo 2π. However, since a rotation by π around the polar angle is already included in the rotation group SO(3), one may choose to work with the simpler equatorial reflection

$$(t, r, \vartheta, \varphi) \mapsto (t', r', \vartheta', \varphi') = (t, r, \pi - \vartheta, \varphi)\,, \tag{4.1.3}$$

which simplifies a number of formulas to be used later.

4.1.2 General spherically symmetric metric-affine geometries

We first give an overview of Lorentzian metrics with spherical symmetry. By using the condition (3.1.14) for the generating vector fields (4.1.1) one arrives at the well-known result that the most general spherically symmetric metric is restricted by the six conditions

$$g_{t\vartheta} = g_{t\varphi} = g_{r\vartheta} = g_{r\varphi} = g_{\vartheta\varphi} = g_{\varphi\varphi} - g_{\vartheta\vartheta}\sin^2\vartheta = 0 \tag{4.1.4}$$

on its components, while the remaining four components are independent. In [76], we chose to parametrize these free components as

$$g_{tt} = -e^{\mathcal{G}_1+\mathcal{G}_2}\cos\mathcal{G}_3\,, \quad g_{rr} = e^{\mathcal{G}_1-\mathcal{G}_2}\cos\mathcal{G}_3\,, \quad g_{tr} = e^{\mathcal{G}_1}\sin\mathcal{G}_3\,, \quad g_{\vartheta\vartheta} = e^{\mathcal{G}_4}\,, \tag{4.1.5}$$

using four free functions $\mathcal{G}_1(t,r),\ldots,\mathcal{G}_4(t,r)$. The reason for this choice is the fact that its determinant then takes the simple form

$$\det g = -e^{2\mathcal{G}_1+2\mathcal{G}_4}\sin^2\vartheta\,, \tag{4.1.6}$$

where the negative sign reflects the fact that the metric has Lorentzian signature, irrespective of the choice of the free functions. It also simplifies the calculation of the inverse, whose independent components are given by

$$g^{tt} = -e^{-\mathcal{G}_1-\mathcal{G}_2}\cos\mathcal{G}_3\,, \quad g^{rr} = e^{-\mathcal{G}_1+\mathcal{G}_2}\cos\mathcal{G}_3\,, \quad g^{tr} = e^{-\mathcal{G}_1}\sin\mathcal{G}_3\,, \quad g^{\vartheta\vartheta} = e^{-\mathcal{G}_4}\,, \tag{4.1.7}$$

and thus take the same form. This simplifies a number of calculations. Finally, we remark that this metric is also invariant under the reflection (4.1.3), and hence under the action of the full spherical symmetry group O(3).

For the affine connection, the condition (3.1.10) for the vector fields (4.1.1) yields 44 conditions in the components. One is thus left with 20 free functions of the coordinates t and r. One possibility to parametrize the non-vanishing components, which is used in [76], is given by

$$\begin{gathered}
\Gamma^t{}_{tt} = \mathcal{C}_1\,, \quad \Gamma^t{}_{tr} = \mathcal{C}_2\,, \quad \Gamma^t{}_{rt} = \mathcal{C}_3\,, \quad \Gamma^t{}_{rr} = \mathcal{C}_4\,, \quad \Gamma^t{}_{\vartheta\vartheta} = \mathcal{C}_9\,,\\
\Gamma^r{}_{tt} = \mathcal{C}_5\,, \quad \Gamma^r{}_{tr} = \mathcal{C}_6\,, \quad \Gamma^r{}_{rt} = \mathcal{C}_7\,, \quad \Gamma^r{}_{rr} = \mathcal{C}_8\,, \quad \Gamma^r{}_{\vartheta\vartheta} = \mathcal{C}_{10}\,,\\
\Gamma^\varphi{}_{t\varphi} = \Gamma^\vartheta{}_{t\vartheta} = \mathcal{C}_{11}\,, \quad \Gamma^\varphi{}_{r\varphi} = \Gamma^\vartheta{}_{r\vartheta} = \mathcal{C}_{12}\,, \quad \Gamma^\varphi{}_{\varphi t} = \Gamma^\vartheta{}_{\vartheta t} = \mathcal{C}_{13}\,, \quad \Gamma^\varphi{}_{\varphi r} = \Gamma^\vartheta{}_{\vartheta r} = \mathcal{C}_{14}\,,\\
\Gamma^\varphi{}_{t\vartheta} = \frac{\mathcal{C}_{15}}{\sin\vartheta}\,, \quad \Gamma^\vartheta{}_{t\varphi} = -\mathcal{C}_{15}\sin\vartheta\,, \quad \Gamma^\varphi{}_{r\vartheta} = \frac{\mathcal{C}_{16}}{\sin\vartheta}\,, \quad \Gamma^\vartheta{}_{r\varphi} = -\mathcal{C}_{16}\sin\vartheta\,,\\
\Gamma^\varphi{}_{\vartheta t} = \frac{\mathcal{C}_{17}}{\sin\vartheta}\,, \quad \Gamma^\vartheta{}_{\varphi t} = -\mathcal{C}_{17}\sin\vartheta\,, \quad \Gamma^\varphi{}_{\vartheta r} = \frac{\mathcal{C}_{18}}{\sin\vartheta}\,, \quad \Gamma^\vartheta{}_{\varphi r} = -\mathcal{C}_{18}\sin\vartheta\,,\\
\Gamma^t{}_{\varphi\vartheta} = \mathcal{C}_{19}\sin\vartheta\,, \quad \Gamma^t{}_{\vartheta\varphi} = -\mathcal{C}_{19}\sin\vartheta\,, \quad \Gamma^r{}_{\varphi\vartheta} = \mathcal{C}_{20}\sin\vartheta\,, \quad \Gamma^r{}_{\vartheta\varphi} = -\mathcal{C}_{20}\sin\vartheta\,,\\
\Gamma^t{}_{\varphi\varphi} = \mathcal{C}_9\sin^2\vartheta\,, \quad \Gamma^r{}_{\varphi\varphi} = \mathcal{C}_{10}\sin^2\vartheta\,, \quad \Gamma^\varphi{}_{\vartheta\varphi} = \Gamma^\varphi{}_{\varphi\vartheta} = \cot\vartheta\,, \quad \Gamma^\vartheta{}_{\varphi\varphi} = -\sin\vartheta\cos\vartheta\,,
\end{gathered} \tag{4.1.8}$$

in terms of 20 free functions $\mathcal{C}_1(t,r),\ldots,\mathcal{C}_{20}(t,r)$, while all other components vanish due to symmetry. In contrast to the metric, however, this connection is in general not invariant under the reflection (4.1.3). Imposing reflection symmetry leads to the additional conditions

$$\mathcal{C}_{15} = \mathcal{C}_{16} = \mathcal{C}_{17} = \mathcal{C}_{18} = \mathcal{C}_{19} = \mathcal{C}_{20} = 0\,, \tag{4.1.9}$$

thus leaving only 14 free functions to determine the most general affine connection invariant under the group O(3).

4.1.3 Characterizing tensors

From the metric (4.1.5) and the affine connection (4.1.8), it is straightforward to calculate the curvature (2.2.7), torsion (2.2.8) and nonmetricity (2.2.9). Further, calculating the Levi-Civita connection (2.2.11), one also finds the contortion (2.2.12) and disformation (2.2.13). We will not give their full expressions here in terms of the parameter functions $\mathcal{G}_1,\ldots,\mathcal{G}_4,\mathcal{C}_1,\ldots,\mathcal{C}_{20}$ given above, as these are rather lengthy, and can be found in full detail in our work [H3]. However, it is worth mentioning a few general properties of these tensors. First, one finds that the non-vanishing, independent components of the most general rotationally symmetric torsion are of the form

$$\begin{aligned} &T^t{}_{tr} = \mathcal{T}_1\,, \quad T^t{}_{\vartheta\varphi} = \mathcal{T}_3 \sin\vartheta\,, \quad T^\vartheta{}_{t\vartheta} = T^\varphi{}_{t\varphi} = \mathcal{T}_5\,, \quad T^\vartheta{}_{t\varphi} = \mathcal{T}_7 \sin\vartheta\,, \quad T^\varphi{}_{t\vartheta} = -\frac{\mathcal{T}_7}{\sin\vartheta}\,, \\ &T^r{}_{tr} = \mathcal{T}_2\,, \quad T^r{}_{\vartheta\varphi} = \mathcal{T}_4 \sin\vartheta\,, \quad T^\vartheta{}_{r\vartheta} = T^\varphi{}_{r\varphi} = \mathcal{T}_6\,, \quad T^\vartheta{}_{r\varphi} = \mathcal{T}_8 \sin\vartheta\,, \quad T^\varphi{}_{r\vartheta} = -\frac{\mathcal{T}_8}{\sin\vartheta}\,, \end{aligned} \tag{4.1.10}$$

while the non-vanishing, independent components of the rotationally symmetric nonmetricity are given by

$$\begin{aligned} Q_{ttt} = \mathcal{Q}_1\,, \quad Q_{trr} = \mathcal{Q}_2\,, \quad Q_{ttr} = \mathcal{Q}_3\,, \quad Q_{\vartheta t\varphi} = -Q_{\varphi t\vartheta} = \mathcal{Q}_{11}\sin\vartheta\,, \\ Q_{rtt} = \mathcal{Q}_5\,, \quad Q_{rrr} = \mathcal{Q}_6\,, \quad Q_{rtr} = \mathcal{Q}_7\,, \quad Q_{\vartheta r\varphi} = -Q_{\varphi r\vartheta} = \mathcal{Q}_{12}\sin\vartheta\,, \\ Q_{t\vartheta\vartheta} = \mathcal{Q}_4\,, \quad Q_{t\varphi\varphi} = \mathcal{Q}_4\sin^2\vartheta\,, \quad Q_{\vartheta t\vartheta} = \mathcal{Q}_9\,, \quad Q_{\varphi t\varphi} = \mathcal{Q}_9\sin^2\vartheta\,, \\ Q_{r\vartheta\vartheta} = \mathcal{Q}_8\,, \quad Q_{r\varphi\varphi} = \mathcal{Q}_8\sin^2\vartheta\,, \quad Q_{\vartheta r\vartheta} = \mathcal{Q}_{10}\,, \quad Q_{\varphi r\varphi} = \mathcal{Q}_{10}\sin^2\vartheta\,, \end{aligned} \tag{4.1.11}$$

and are therefore expressed in terms of parameter functions $\mathcal{T}_1(t,r),\ldots,\mathcal{T}_8(t,r)$ for the torsion, as well as $\mathcal{Q}_1(t,r),\ldots,\mathcal{Q}_{12}(t,r)$ for the nonmetricity. This general form can also be obtained independently of the metric and affine connection we found previously, by imposing that the Lie derivative of the torsion and nonmetricity tensors with respect to the vector fields (4.1.1) vanishes. Via the formulas (2.2.8) and (2.2.9), the new parameter functions $\mathcal{T}_1,\ldots,\mathcal{T}_8,\mathcal{Q}_1,\ldots,\mathcal{Q}_{12}$ can be expressed through the original parameter functions $\mathcal{G}_1,\ldots,\mathcal{G}_4,\mathcal{C}_1,\ldots,\mathcal{C}_{20}$. Further, from the torsion, nonmetricity and metric one obtains the contortion

$$\begin{aligned} K_{t\varphi\vartheta} = -K_{t\vartheta\varphi} &= \frac{1}{2}e^{\mathcal{G}_1}\left(\mathcal{T}_4\sin\mathcal{G}_3 - \mathcal{T}_3 e^{\mathcal{G}_2}\cos\mathcal{G}_3\right)\sin\vartheta\,, \\ K_{r\varphi\vartheta} = -K_{r\vartheta\varphi} &= \frac{1}{2}e^{\mathcal{G}_1}\left(\mathcal{T}_3\sin\mathcal{G}_3 + \mathcal{T}_4 e^{-\mathcal{G}_2}\cos\mathcal{G}_3\right)\sin\vartheta\,, \\ K_{\vartheta\varphi t} &= \frac{1}{2}\left[2\mathcal{T}_7 e^{\mathcal{G}_4} + \left(\mathcal{T}_4\sin\mathcal{G}_3 - \mathcal{T}_3 e^{\mathcal{G}_2}\cos\mathcal{G}_3\right)e^{\mathcal{G}_1}\right]\sin\vartheta\,, \\ K_{\vartheta\varphi r} &= \frac{1}{2}\left[2\mathcal{T}_8 e^{\mathcal{G}_4} + \left(\mathcal{T}_3\sin\mathcal{G}_3 + \mathcal{T}_4 e^{-\mathcal{G}_2}\cos\mathcal{G}_3\right)e^{\mathcal{G}_1}\right]\sin\vartheta\,, \\ K_{trt} = e^{\mathcal{G}_1}\left(\mathcal{T}_2\sin\mathcal{G}_3 - \mathcal{T}_1 e^{\mathcal{G}_2}\cos\mathcal{G}_3\right)\,, &\quad K_{t\vartheta\vartheta} = \frac{K_{t\varphi\varphi}}{\sin^2\vartheta} = e^{\mathcal{G}_4}\mathcal{T}_5\,, \\ K_{trr} = e^{\mathcal{G}_1}\left(\mathcal{T}_1\sin\mathcal{G}_3 + \mathcal{T}_2 e^{-\mathcal{G}_2}\cos\mathcal{G}_3\right)\,, &\quad K_{r\vartheta\vartheta} = \frac{K_{r\varphi\varphi}}{\sin^2\vartheta} = e^{\mathcal{G}_4}\mathcal{T}_6\,, \end{aligned} \tag{4.1.12}$$

as well as the disformation

$$\begin{aligned} &L_{ttt} = -\frac{1}{2}\mathcal{Q}_1\,, \qquad L_{trr} = \frac{1}{2}\mathcal{Q}_2 - \mathcal{Q}_7\,, \qquad L_{t\vartheta\vartheta} = \frac{L_{t\varphi\varphi}}{\sin^2\vartheta} = \frac{1}{2}\mathcal{Q}_4 - \mathcal{Q}_9\,, \\ &L_{rrr} = -\frac{1}{2}\mathcal{Q}_6\,, \qquad L_{rtt} = \frac{1}{2}\mathcal{Q}_5 - \mathcal{Q}_3\,, \qquad L_{r\vartheta\vartheta} = \frac{L_{r\varphi\varphi}}{\sin^2\vartheta} = \frac{1}{2}\mathcal{Q}_8 - \mathcal{Q}_{10}\,, \\ &L_{ttr} = -\frac{1}{2}\mathcal{Q}_5\,, \qquad L_{\vartheta t\varphi} = -L_{\varphi t\vartheta} = \mathcal{Q}_{11}\sin\vartheta\,, \qquad L_{\vartheta t\vartheta} = \frac{L_{\varphi t\varphi}}{\sin^2\theta} = -\frac{1}{2}\mathcal{Q}_4\,, \\ &L_{rtr} = -\frac{1}{2}\mathcal{Q}_2\,, \qquad L_{\vartheta r\varphi} = -L_{\varphi r\vartheta} = \mathcal{Q}_{12}\sin\vartheta\,, \qquad L_{\vartheta r\vartheta} = \frac{L_{\varphi r\varphi}}{\sin^2\theta} = -\frac{1}{2}\mathcal{Q}_8\,. \end{aligned} \tag{4.1.13}$$

One may the use the decomposition (2.2.10) to express the parameter functions $\mathcal{C}_1,\ldots,\mathcal{C}_{20}$ in terms of the parameter functions $\mathcal{G}_1,\ldots,\mathcal{G}_4,\mathcal{T}_1,\ldots,\mathcal{T}_8,\mathcal{Q}_1,\ldots,\mathcal{Q}_{12}$. Hence, in the presence of the metric parameter functions $\mathcal{G}_1,\ldots,\mathcal{G}_4$, one obtains a one-to-one correspondence between the parametrizations

$$\mathcal{C}_1, \ldots, \mathcal{C}_{20} \quad \leftrightsquigarrow \quad \mathcal{T}_1, \ldots, \mathcal{T}_8, \mathcal{Q}_1, \ldots, \mathcal{Q}_{12}\,, \tag{4.1.14}$$

which is given by a linear transformation, due to the linearity of the defining equations of torsion (2.2.8) and nonmetricity (2.2.9), as well as the corresponding inverse relations. Finally, we remark that the condition (4.1.9) is equivalently expressed as

$$\mathcal{T}_3 = \mathcal{T}_4 = \mathcal{T}_7 = \mathcal{T}_8 = \mathcal{Q}_{11} = \mathcal{Q}_{12} = 0 \tag{4.1.15}$$

in terms of the torsion and nonmetricity components.

4.1.4 Special cases

Based on the parametrization of the connection by the functions $\mathcal{T}_1, \ldots, \mathcal{T}_8$ associated to torsion and $\mathcal{Q}_1, \ldots, \mathcal{Q}_{12}$ associated to nonmetricity, it is now straightforward to derive the most general symmetric (torsion-free) and metric-compatible metric-affine geometries, by demanding that the corresponding parameter functions vanish. These conditions that translate to a simple set of conditions on the original parameter functions $\mathcal{C}_1, \ldots, \mathcal{C}_{20}$, due to the linearity of the relations (4.1.14). The condition imposed by vanishing curvature, however, is significantly more involved, since by its definition (2.2.7), it depends quadratically on the connection coefficients, as well as linearly on their derivatives. The resulting non-linear partial differential equations arising from the condition $R^{\rho}{}_{\sigma\mu\nu} \equiv 0$ are highly non-trivial. Instead of solving them directly, one can use the fact that the existence of a flat connection on a simply connected manifold allows to construct a global coframe $\tilde{\theta}^A{}_\mu$, whose constituting covectors are obtained by parallel transport with respect to the flat and hence path-independent connection. From this construction, which implies that the coframe components satisfy the condition

$$0 = \partial_\mu \tilde{\theta}^A{}_\nu - \Gamma^\rho{}_{\nu\mu}\tilde{\theta}^A{}_\rho\,, \tag{4.1.16}$$

then follows that the connection coefficients are given by the Weitzenböck connection (2.2.23), with the tetrad $\tilde{\theta}^A{}_\mu$ in place of $\theta^A{}_\mu$. The most general tetrad which obeys the symmetry under rotations was found in our work [H2] and takes the form

$$\tilde{\theta}^0 = \mathcal{F}_1 \cosh \mathcal{F}_3 \mathrm{d}t + \mathcal{F}_2 \sinh \mathcal{F}_4 \mathrm{d}r\,, \tag{4.1.17a}$$

$$\begin{aligned}\tilde{\theta}^1 &= \sin\vartheta\cos\varphi(\mathcal{F}_1 \sinh \mathcal{F}_3 \mathrm{d}t + \mathcal{F}_2 \cosh \mathcal{F}_4 \mathrm{d}r)\\ &\quad + \mathcal{F}_5\left[(\cos\mathcal{F}_6\cos\vartheta\cos\varphi - \sin\mathcal{F}_6\sin\varphi)\mathrm{d}\vartheta - \sin\vartheta(\cos\mathcal{F}_6\sin\varphi + \sin\mathcal{F}_6\cos\vartheta\cos\varphi)\mathrm{d}\varphi\right]\,,\end{aligned} \tag{4.1.17b}$$

$$\begin{aligned}\tilde{\theta}^2 &= \sin\vartheta\sin\varphi(\mathcal{F}_1 \sinh \mathcal{F}_3 \mathrm{d}t + \mathcal{F}_2 \cosh \mathcal{F}_4 \mathrm{d}r)\\ &\quad + \mathcal{F}_5\left[(\cos\mathcal{F}_6\cos\vartheta\sin\varphi + \sin\mathcal{F}_6\cos\varphi)\mathrm{d}\vartheta + \sin\vartheta(\cos\mathcal{F}_6\cos\varphi - \sin\mathcal{F}_6\cos\vartheta\sin\varphi)\mathrm{d}\varphi\right]\,,\end{aligned} \tag{4.1.17c}$$

$$\tilde{\theta}^3 = \cos\vartheta(\mathcal{F}_1 \sinh \mathcal{F}_3 \mathrm{d}t + \mathcal{F}_2 \cosh \mathcal{F}_4 \mathrm{d}r) + \mathcal{F}_5\left[-\cos\mathcal{F}_6\sin\vartheta\mathrm{d}\vartheta + \sin\mathcal{F}_6\sin^2\vartheta\mathrm{d}\varphi\right]\,. \tag{4.1.17d}$$

in terms of the free functions $\mathcal{F}_1(t,r), \ldots, \mathcal{F}_6(t,r)$. Further imposing also invariance under reflections on the resulting Weitzenböck connection (2.2.23), one obtains the additional restriction $\mathcal{F}_6 \equiv 0$, so that in this case one is left with only the free functions $\mathcal{F}_1, \ldots, \mathcal{F}_5$.

4.2 Cosmological symmetry in teleparallel geometry

In our work [H2], where we proposed a general concept of symmetry in teleparallel geometry, we gave examples of cosmologically symmetric teleparallel spacetimes, i.e., homogeneous and isotropic teleparallel geometries, which are therefore invariant under both rotations and translations. This immediately led to the question whether further examples can be found, and how to obtain an exhaustive classification of cosmologically symmetric teleparallel geometries. We answered these questions in our work [H4], which can be understood as a continuation of the work [H3] pre presented in the previous section, and which we now summarize. We start with a brief summary of the conditions of cosmological symmetry and the overall solution procedure in section 4.2.1. We then summarize several steps of this procedure and the obtained results. In section 4.2.2, we show how to obtain a general metric-affine geometry with cosmological symmetry. This is further restricted to a metric teleparallel geometry in section 4.2.3. The corresponding formulation in terms of a tetrad and spin connection is given in section 4.2.4. An alternative approach based on the representation theory of the symmetry group, which immediately leads the tetrad in the Weitzenböck gauge, is shown in section 4.2.5. Finally, in section 4.2.6, we show an example application by deriving the cosmological field equations of a general class of teleparallel gravity theories.

4.2.1 General procedure

From the most general spherically symmetric metric-affine geometry discussed in the previous section, more restricted classes of geometries can be derived by imposing further conditions. We now impose the following two conditions:

1. Instead of general metric-affine geometry, only (metric) teleparallel geometry is considered. Hence, the conditions of vanishing curvature, $R^{\mu}{}_{\nu\rho\sigma} = 0$, and vanishing nonmetricity, $Q_{\rho\mu\nu} = 0$, are imposed.

2. In addition to the generators (4.1.1) of rotations, also symmetry under three translation generators is imposed. Using the same spherical coordinate system as above, these generators take the form

$$T_1 = \chi\sin\vartheta\cos\varphi\partial_r + \frac{\chi}{r}\cos\vartheta\cos\varphi\partial_\vartheta - \frac{\chi\sin\varphi}{r\sin\vartheta}\partial_\varphi\,, \tag{4.2.1a}$$

$$T_2 = \chi\sin\vartheta\sin\varphi\partial_r + \frac{\chi}{r}\cos\vartheta\sin\varphi\partial_\vartheta + \frac{\chi\cos\varphi}{r\sin\vartheta}\partial_\varphi\,, \tag{4.2.1b}$$

$$T_3 = \chi\cos\vartheta\partial_r - \frac{\chi}{r}\sin\vartheta\partial_\vartheta\,, \tag{4.2.1c}$$

   where we used the abbreviation $\chi = \sqrt{1-(ur)^2}$, and u is an arbitrary, real or imaginary constant. Note that it is more common to introduce a real parameter $k = u^2$ instead, whose sign corresponds to the sign of the curvature of the spatial hypersurfaces of the resulting geometry, and to restrict its values to $k \in \{-1, 0, 1\}$. For our purposes, however, it turns out to be more useful to introduce a continuous parameter u, as we will see later.

The most general geometry which satisfies these two additional conditions is derived in our work [H4]. Three different approaches are used, which can essentially be divided into two classes:

1. The most straightforward approach is to start from the most general spherically symmetric metric-affine geometry, which is given by the metric (4.1.5) and affine connection with coefficients (4.1.8), and to further impose the two conditions given above, in order to restrict the parameter functions $\mathcal{G}_1, \ldots, \mathcal{G}_4$ and $\mathcal{C}_1, \ldots, \mathcal{C}_{20}$. Depending on the order in which they are imposed, one arrives at one of two intermediate steps: either teleparallel geometry with spherical symmetry, or metric-affine geometry with cosmological symmetry. Further, one may divide this approach into smaller subclasses, by either considering directly the coefficients $\Gamma^{\mu}{}_{\nu\rho}$ of the affine connection, or by using the decomposition (2.2.10) into the Levi-Civita connection, contortion and disformation. The result obtained in all these cases is a metric-affine geometry, expressed in terms of a cosmologically symmetric metric $g_{\mu\nu}$ and flat, metric-compatible affine connection $\Gamma^{\mu}{}_{\nu\rho}$. In order to express this geometry in terms of a tetrad and spin connection, one then further needs to chose a tetrad representing the metric, and can derive the spin connection from the tetrad postulate (2.2.29).

2. The second approach makes use of the notion of symmetry of teleparallel geometries discussed in section 3.2. This approach has the advantage that one obtains the teleparallel geometry directly in terms of the tetrad and spin connection, which may be chosen a priori to vanish by working in the Weitzenböck gauge. This means that the condition of vanishing curvature and nonmetricity is satisfied already by the choice of the variables which describe the geometry, and no further equations must be solved to impose this condition. However, this convenience comes at the cost of the necessity to chose a homomorphism from the symmetry group to the Lorentz group, which enters the symmetry condition (3.2.1) in the Weitzenböck gauge. While it is not difficult to find examples for such homomorphisms, as we have shown in our work [H2], in order to determine the most general class of cosmologically symmetric teleparallel geometries, one must determine all possible homomorphisms. This can be achieved by realizing that any such homomorphism constitutes a four-dimensional representation of the symmetry group, which preserves a bilinear form of Lorentzian signature. By using representation theory, all such representations can be obtained.

In the following sections, we give an overview of the single steps mentioned in the two approaches above, and show how they can be applied in order to determine the most general teleparallel geometry with cosmological symmetry.

4.2.2 Cosmologically symmetric metric-affine geometry

We start from the metric-affine approach. The first step to consider here is to restrict the class of spherically symmetric metric-affine geometries given in section 4.1 to cosmological symmetry, by imposing the following conditions.

1. *Spherical to cosmological metric*: Starting from the spherically symmetric metric (4.1.5) and imposing the symmetry condition (3.1.14) for the translation generators (4.2.1), one finds that the most general metric with cosmological symmetry is given by

$$g_{tt} = -\mathcal{N}^2 \,, \quad g_{rr} = \frac{\mathcal{A}^2}{\chi^2} \,, \quad g_{\vartheta\vartheta} = \mathcal{A}^2 r^2 \,, \quad g_{\varphi\varphi} = g_{\vartheta\vartheta} \sin^2\vartheta \,, \tag{4.2.2}$$

where $\mathcal{N}(t)$ is the lapse function and $\mathcal{A}(t)$ is the scale factor. This is of course the well-known Friedmann-Lemaître-Robertson-Walker metric, in arbitrary time parametrization.

2. *Spherical to cosmological affine connection*: As for the metric in the previous step, one may impose the symmetry condition (3.1.10) with the translation generators (4.2.1) on the spherically symmetric affine connection (4.1.8). One finds that the most general cosmologically symmetric affine connection is given by [76]

$$\begin{gathered}
\Gamma^t{}_{tt} = \mathcal{K}_1 \,, \quad \Gamma^t{}_{rr} = \frac{\mathcal{K}_2}{\chi^2} \,, \quad \Gamma^t{}_{\vartheta\vartheta} = \mathcal{K}_2 r^2 \,, \quad \Gamma^t{}_{\varphi\varphi} = \mathcal{K}_2 r^2 \sin^2\vartheta \,, \\
\Gamma^\vartheta{}_{r\vartheta} = \Gamma^\vartheta{}_{\vartheta r} = \Gamma^\varphi{}_{r\varphi} = \Gamma^\varphi{}_{\varphi r} = \frac{1}{r} \,, \quad \Gamma^\varphi{}_{\vartheta\varphi} = \Gamma^\varphi{}_{\varphi\vartheta} = \cot\vartheta \,, \quad \Gamma^\vartheta{}_{\varphi\varphi} = -\sin\vartheta\cos\vartheta \,, \\
\Gamma^r{}_{\vartheta\vartheta} = -r\chi^2 \,, \quad \Gamma^r{}_{\varphi\varphi} = -r\chi^2\sin^2\vartheta \,, \quad \Gamma^r{}_{\varphi\vartheta} = -\Gamma^r{}_{\vartheta\varphi} = \mathcal{K}_5 r^2 \chi \sin\vartheta \,, \\
\Gamma^r{}_{tr} = \Gamma^\vartheta{}_{t\vartheta} = \Gamma^\varphi{}_{t\varphi} = \mathcal{K}_3 \,, \quad \Gamma^r{}_{rt} = \Gamma^\vartheta{}_{\vartheta t} = \Gamma^\varphi{}_{\varphi t} = \mathcal{K}_4 \,, \quad \Gamma^r{}_{rr} = \frac{u^2 r}{\chi^2} \,, \\
\Gamma^\vartheta{}_{r\varphi} = -\Gamma^\vartheta{}_{\varphi r} = \frac{\mathcal{K}_5 \sin\vartheta}{\chi} \,, \quad \Gamma^\varphi{}_{r\vartheta} = -\Gamma^\varphi{}_{\vartheta r} = -\frac{\mathcal{K}_5}{\chi\sin\vartheta} \,,
\end{gathered} \tag{4.2.3}$$

and thus depends on five arbitrary functions $\mathcal{K}_1(t), \ldots, \mathcal{K}_5(t)$ of time.

The geometry obtained after these two steps is the most general metric-affine geometry with cosmological symmetry. In analogy to the spherically symmetric geometry, it is instructive to study its curvature, torsion and nonmetricity, before proceeding to more restricted geometries. For this purpose it turns out to be helpful to decompose the metric in the form

$$g_{\mu\nu} = -n_\mu n_\nu + h_{\mu\nu} \tag{4.2.4}$$

into the hypersurface conormal n_μ and spatial metric $h_{\mu\nu}$, whose non-vanishing components are given by

$$n_t = -\mathcal{N} \,, \quad h_{rr} = \frac{\mathcal{A}^2}{\chi^2} \,, \quad h_{\vartheta\vartheta} = \mathcal{A}^2 r^2 \,, \quad h_{\varphi\varphi} = h_{\vartheta\vartheta}\sin^2\vartheta \,. \tag{4.2.5}$$

Further, one derives from the Levi-Civita tensor of the spacetime metric $g_{\mu\nu}$, which is the totally antisymmetric tensor with normalized components

$$\epsilon_{tr\vartheta\varphi} = \frac{\mathcal{N}\mathcal{A}^3 r^2 \sin\vartheta}{\chi} \,, \quad \epsilon^{tr\vartheta\varphi} = -\frac{\chi}{\mathcal{N}\mathcal{A}^3 r^2 \sin\vartheta} \,, \tag{4.2.6}$$

its contraction with the unit normal,

$$\varepsilon_{\mu\nu\rho} = n^\sigma \epsilon_{\sigma\mu\nu\rho} \,, \quad \epsilon_{\mu\nu\rho\sigma} = 4\varepsilon_{[\mu\nu\rho}n_{\sigma]} \,. \tag{4.2.7}$$

This is the Levi-Civita tensor of the induced metric $h_{\mu\nu}$, and its components are normalized as

$$\varepsilon_{r\vartheta\varphi} = \frac{\mathcal{A}^3 r^2 \sin\vartheta}{\chi} \,, \quad \varepsilon^{r\vartheta\varphi} = \frac{\chi}{\mathcal{A}^3 r^2 \sin\vartheta} \,. \tag{4.2.8}$$

The reason for introducing these objects becomes clear below. One finds that the curvature of the connection (4.2.3) can now be written in the form

$$\begin{aligned}
R^\mu{}_{\nu\rho\sigma} &= 2\frac{\mathcal{K}_3(\mathcal{K}_4 - \mathcal{K}_1) + \dot{\mathcal{K}}_3}{\mathcal{N}^2} n_\nu n_{[\rho} h^\mu_{\sigma]} + 2\frac{\mathcal{K}_2(\mathcal{K}_4 - \mathcal{K}_1) - \dot{\mathcal{K}}_2}{\mathcal{A}^2} n^\mu n_{[\rho} h_{\sigma]\nu} \\
&+ 2\frac{\mathcal{K}_2\mathcal{K}_5\mathcal{N}}{\mathcal{A}^3} n^\mu \varepsilon_{\nu\rho\sigma} - 2\frac{\mathcal{K}_3\mathcal{K}_5}{\mathcal{N}\mathcal{A}} n_\nu \varepsilon^\mu{}_{\rho\sigma} - 2\frac{\dot{\mathcal{K}}_5}{\mathcal{N}\mathcal{A}} \varepsilon^\mu{}_{\nu[\rho} n_{\sigma]} + 2\frac{u^2 + \mathcal{K}_2\mathcal{K}_3 - \mathcal{K}_5^2}{\mathcal{A}^2} h^\mu_{[\rho} h_{\sigma]\nu} \,,
\end{aligned} \tag{4.2.9}$$

where a dot denotes the derivative with respect to the time coordinate t. In a similar fashion, one may express the torsion

$$T^\mu{}_{\nu\rho} = 2\mathcal{T}_1 h^\mu_{[\nu} n_{\rho]} + 2\mathcal{T}_2 \varepsilon^\mu{}_{\nu\rho} \tag{4.2.10}$$

and the nonmetricity

$$Q_{\rho\mu\nu} = 2\mathcal{Q}_1 n_\rho n_\mu n_\nu + 2\mathcal{Q}_2 n_\rho h_{\mu\nu} + 2\mathcal{Q}_3 h_{\rho(\mu} n_{\nu)} \tag{4.2.11}$$

of the cosmologically symmetric metric-affine geometry, where we have introduced the convenient parametrization in terms of the two torsion scalars

$$\mathcal{T}_1 = \frac{\mathcal{K}_4 - \mathcal{K}_3}{\mathcal{N}}, \quad \mathcal{T}_2 = \frac{\mathcal{K}_5}{\mathcal{A}}, \tag{4.2.12}$$

as well as the three nonmetricity scalars

$$\mathcal{Q}_1 = \frac{\dot{\mathcal{N}}}{\mathcal{N}^2} - \frac{\mathcal{K}_1}{\mathcal{N}}, \quad \mathcal{Q}_2 = \frac{1}{\mathcal{N}}\left(\mathcal{K}_4 - \frac{\dot{\mathcal{A}}}{\mathcal{A}}\right), \quad \mathcal{Q}_3 = \frac{\mathcal{K}_3}{\mathcal{N}} - \frac{\mathcal{K}_2\mathcal{N}}{\mathcal{A}^2}. \tag{4.2.13}$$

Note that these five scalars, which are functions of time t, yield a parametrization of the connection (4.2.3), which is equivalent to the five functions functions $\mathcal{K}_1, \ldots, \mathcal{K}_5$ we introduced earlier. This can be seen by explicitly inverting the relations above, or by using the decomposition (2.2.10) with the contortion

$$K_{\mu\nu\rho} = 2\mathcal{T}_1 h_{\rho[\mu} n_{\nu]} - \mathcal{T}_2 \varepsilon_{\mu\nu\rho}, \tag{4.2.14}$$

and the disformation

$$L_{\rho\mu\nu} = -\mathcal{Q}_1 n_\rho n_\mu n_\nu + (\mathcal{Q}_2 - \mathcal{Q}_3) n_\rho h_{\mu\nu} - 2\mathcal{Q}_2 h_{\rho(\mu} n_{\nu)}, \tag{4.2.15}$$

which are readily obtained from the torsion and nonmetricity given above.

4.2.3 Restriction to teleparallel geometry

In order to obtain a teleparallel geometry with cosmological symmetry, one further needs to impose the conditions of vanishing nonmetricity and curvature. Since these are two independent conditions, which can be applied in any order, it is instructive to study both possibilities. If one imposes only vanishing nonmetricity, given by the three scalar functions (4.2.13), one obtains the conditions

$$\mathcal{K}_1\mathcal{N} - \dot{\mathcal{N}} = \mathcal{K}_4\mathcal{A} - \dot{\mathcal{A}} = \mathcal{K}_2\mathcal{N}^2 - \mathcal{K}_3\mathcal{A}^2 = 0, \tag{4.2.16}$$

which fully determine the connection coefficients $\mathcal{K}_1$ and $\mathcal{K}_4$, as well as the ratio between $\mathcal{K}_2$ and $\mathcal{K}_3$. Also note that this ratio is fixed to be positive, $\mathcal{A}^2/\mathcal{N}^2$, which is a consequence of the Lorentz signature of the metric.

If one imposes only vanishing curvature instead, one arrives at the conditions

$$\dot{\mathcal{K}}_5 = \mathcal{K}_2\mathcal{K}_5 = \mathcal{K}_3\mathcal{K}_5 = u^2 + \mathcal{K}_2\mathcal{K}_3 - \mathcal{K}_5^2 = \mathcal{K}_3(\mathcal{K}_4 - \mathcal{K}_1) + \dot{\mathcal{K}}_3 = \mathcal{K}_2(\mathcal{K}_4 - \mathcal{K}_1) - \dot{\mathcal{K}}_2 = 0. \tag{4.2.17}$$

To study the possible solutions of this system of equations, it is useful to first distinguish the following two cases:

1. $u = 0$: In this case we have the condition $\mathcal{K}_2\mathcal{K}_3 = \mathcal{K}_5^2$, so either both sides are vanishing or non-vanishing. However, from $\mathcal{K}_2\mathcal{K}_5 = \mathcal{K}_3\mathcal{K}_5 = 0$ follows that $\mathcal{K}_5 = 0$ or $\mathcal{K}_2 = \mathcal{K}_3 = 0$. Hence, the only option is $\mathcal{K}_5 = \mathcal{K}_2\mathcal{K}_3 = 0$. Therefore, at least one of $\mathcal{K}_2$ or $\mathcal{K}_3$ must vanish:

   (a) $\mathcal{K}_2 = \mathcal{K}_3 = 0$: In this case the remaining equations are satisfied. $\mathcal{K}_1$ and $\mathcal{K}_4$ are the only parameters left, which are arbitrary and unconstrained.

   (b) $\mathcal{K}_2 \neq 0$: In this case, $\mathcal{K}_2$ is a free function, while only one of $\mathcal{K}_1$ and $\mathcal{K}_4$ is left undetermined, and their difference satisfies

   $$\mathcal{K}_4 - \mathcal{K}_1 = \frac{\dot{\mathcal{K}}_2}{\mathcal{K}_2}\,. \tag{4.2.18}$$

   (c) $\mathcal{K}_3 \neq 0$: As in the case before, but now $\mathcal{K}_3$ is a free function, and determines the difference of $\mathcal{K}_1$ and $\mathcal{K}_4$ through

   $$\mathcal{K}_4 - \mathcal{K}_1 = -\frac{\dot{\mathcal{K}}_3}{\mathcal{K}_3}\,. \tag{4.2.19}$$

2. $u \neq 0$: Here we can distinguish the following two cases:

   (a) $\mathcal{K}_5 \neq 0$: From $\mathcal{K}_2\mathcal{K}_5 = \mathcal{K}_3\mathcal{K}_5 = 0$ follows $\mathcal{K}_2 = \mathcal{K}_3 = 0$. Hence, $\mathcal{K}_5 = \pm u$, and the remaining equations are satisfied. $\mathcal{K}_1$ and $\mathcal{K}_4$ are left undetermined.

   (b) $\mathcal{K}_5 = 0$: In this case one has $\mathcal{K}_2\mathcal{K}_3 = -u^2 \neq 0$ and so both must be non-zero and inversely proportional, so that only one of them can be chosen arbitrarily. This further implies

   $$0 = \dot{\mathcal{K}}_2\mathcal{K}_3 + \mathcal{K}_2\dot{\mathcal{K}}_3\,, \tag{4.2.20}$$

   and so

   $$\mathcal{K}_4 - \mathcal{K}_1 = \frac{\dot{\mathcal{K}}_2}{\mathcal{K}_2} = -\frac{\dot{\mathcal{K}}_3}{\mathcal{K}_3}\,, \tag{4.2.21}$$

   so that the remaining equations consistently determine $\mathcal{K}_4 - \mathcal{K}_1$, while their sum is left undetermined.

We see that in all cases we are left with two free functions, which parametrize the chosen branch of solutions, and in terms of which all other parameter functions are determined.

To obtain a metric teleparallel geometry, we must impose both conditions of vanishing nonmetricity and curvature simultaneously. From the condition (4.2.16) the two parameter functions $\mathcal{K}_1$ and $\mathcal{K}_4$ are uniquely determined as

$$\mathcal{K}_1 = \frac{\dot{\mathcal{N}}}{\mathcal{N}}\,, \quad \mathcal{K}_4 = \frac{\dot{\mathcal{A}}}{\mathcal{A}}\,. \tag{4.2.22}$$

To solve the flatness (4.2.17) condition, it is again useful to first distinguish two cases:

1. $u = 0$: Flatness implies $\mathcal{K}_5 = 0$ and at least one of $\mathcal{K}_2$ or $\mathcal{K}_3$ must also vanish. From metric compatibility follows that if one of them vanishes, so does the other, and so the only possibility is $\mathcal{K}_2 = \mathcal{K}_3 = 0$.

2. $u \neq 0$: Flatness leaves two options:

   (a) $\mathcal{K}_5 = \pm u$: This implies $\mathcal{K}_2 = \mathcal{K}_3 = 0$, and so all functions are fixed.

(b) $\mathcal{K}_5 = 0$: Now one has $\mathcal{K}_2\mathcal{K}_3 = -u^2$. Together with metric compatibility this yields

$$\mathcal{K}_2 = \pm iu\frac{\mathcal{A}}{\mathcal{N}}, \quad \mathcal{K}_3 = \pm iu\frac{\mathcal{N}}{\mathcal{A}}, \tag{4.2.23}$$

where the same sign must be chosen for both terms, so that also in this case all parameter functions are fully determined.

One sees that the formulas derived for the two branches with $u \neq 0$ also hold in the case $u = 0$, where they reduce to $\mathcal{K}_2 = \mathcal{K}_3 = \mathcal{K}_5 = 0$ for both branches. Hence, one finds that for $u = 0$, all parameter functions in the connection are fully determined, while for $u \neq 0$, one has either $\mathcal{K}_5$ or $\mathcal{K}_2$ and $\mathcal{K}_3$ non-vanishing and determined by u up to a sign.

Now it also becomes clear why we have chosen the continuous, real or imaginary parameter u instead of the more common discrete parameter $k = u^2 \in \{-1, 0, 1\}$, as mentioned in section 4.2.1. We see that u explicitly appears in the connection coefficients, and so we avoid using a square root. Also note that depending on the sign of u^2, only one of the solution branches for $u \neq 0$ yields a connection with real coefficients, while the other branch becomes complex; we will discuss the implications of this finding later. Finally, we see that both branches possess a common continuous limit $u \to 0$, which is the reason for considering a continuous parameter. This allows us to lift the distinction between the cases $u = 0$ and $u \neq 0$ from now on, and consider the former as a limiting case of the latter.

Finally, it is helpful to calculate the torsion tensor for the two branches of teleparallel geometries we found above. Using the form (4.2.12), one finds that for the branch with $\mathcal{K}_5 = 0$ one has

$$\mathcal{T}_1 = \frac{\dot{\mathcal{A}}}{\mathcal{N}\mathcal{A}} \pm \frac{iu}{\mathcal{A}}, \quad \mathcal{T}_2 = 0, \tag{4.2.24}$$

so that the only irreducible component of the torsion is its vectorial part, while for $\mathcal{K}_5 \neq 0$ one finds

$$\mathcal{T}_1 = \frac{\dot{\mathcal{A}}}{\mathcal{N}\mathcal{A}}, \quad \mathcal{T}_2 = \pm\frac{u}{\mathcal{A}}, \tag{4.2.25}$$

which in addition has also axial torsion. Hence, it makes sense to name these two branches the “vector” and “axial” branch, respectively. Note that these two scalars fully determine the connection coefficients through the contortion and the decomposition (2.2.10). For later convenience, it is useful to introduce the rescaled parameter functions

$$\mathfrak{v} = \mathcal{A}\mathcal{T}_1, \quad \mathfrak{a} = \mathcal{A}\mathcal{T}_2, \tag{4.2.26}$$

as well as the conformal Hubble parameter

$$\mathcal{H} = \frac{\partial_t\mathcal{A}}{\mathcal{N}}. \tag{4.2.27}$$

Then the two branches are characterized by

$$\mathfrak{v} = \mathcal{H} \pm iu, \quad \mathfrak{a} = 0 \tag{4.2.28}$$

and

$$\mathfrak{v} = \mathcal{H}, \quad \mathfrak{a} = \pm u, \tag{4.2.29}$$

respectively.

4.2.4 Tetrad and spin connection

As discussed in section 2.2.3, it is more common to express a teleparallel geometry in terms of a tetrad and a spin connection, instead of the metric-affine formulation in terms of a metric and affine connection. The most obvious possible choice of a tetrad for the Friedmann-Lemaître-Robertson-Walker metric (4.2.2) is the diagonal tetrad

$$\theta'^0 = \mathcal{N}\mathrm{d}t\,, \quad \theta'^1 = \frac{\mathcal{A}}{\chi}\mathrm{d}r\,, \quad \theta'^2 = \mathcal{A}r\mathrm{d}\vartheta\,, \quad \theta'^3 = \mathcal{A}r\sin\vartheta\mathrm{d}\varphi\,, \tag{4.2.30}$$

where we use a prime here to denote the diagonal gauge, to distinguish it from the Weitzenböck gauge used later. Given this tetrad and an affine connection with coefficients $\Gamma^{\mu}{}_{\nu\rho}$, the corresponding spin connection is uniquely determined from the tetrad postulate (2.2.29). For the cosmologically symmetric affine connection (4.2.3) and the diagonal tetrad (4.2.30), the spin connection is given by

$$\begin{gathered}
\omega'^0{}_{0t} = \mathcal{K}_1 - \frac{\dot{\mathcal{N}}}{\mathcal{N}}\,, \quad \omega'^1{}_{1t} = \omega'^2{}_{2t} = \omega'^3{}_{3t} = \mathcal{K}_4 - \frac{\dot{\mathcal{A}}}{\mathcal{A}}\,,\\
\omega'^0{}_{1r} = \frac{\mathcal{N}\mathcal{K}_2}{\mathcal{A}\chi}\,, \quad \omega'^0{}_{2\vartheta} = \frac{\mathcal{N}\mathcal{K}_2 r}{\mathcal{A}}\,, \quad \omega'^0{}_{3\varphi} = \frac{\mathcal{N}\mathcal{K}_2 r\sin\vartheta}{\mathcal{A}}\,,\\
\omega'^1{}_{0r} = \frac{\mathcal{A}\mathcal{K}_3}{\mathcal{N}\chi}\,, \quad \omega'^2{}_{0\vartheta} = \frac{\mathcal{A}\mathcal{K}_3 r}{\mathcal{N}}\,, \quad \omega'^3{}_{0\varphi} = \frac{\mathcal{A}\mathcal{K}_3 r\sin\vartheta}{\mathcal{N}}\,,\\
\omega'^3{}_{2r} = -\omega'^2{}_{3r} = \frac{\mathcal{K}_5}{\chi}\,, \quad \omega'^1{}_{3\vartheta} = -\omega'^3{}_{1\vartheta} = \mathcal{K}_5 r\,, \quad \omega'^2{}_{1\varphi} = -\omega'^1{}_{2\varphi} = \mathcal{K}_5 r\sin\vartheta\,,\\
\omega'^2{}_{1\vartheta} = -\omega'^1{}_{2\vartheta} = \chi\,, \quad \omega'^3{}_{1\varphi} = -\omega'^1{}_{3\varphi} = \chi\sin\vartheta\,, \quad \omega'^3{}_{2\varphi} = -\omega'^2{}_{3\varphi} = \cos\vartheta\,.
\end{gathered} \tag{4.2.31}$$

Note that this spin connection is neither flat, nor antisymmetric in general; imposing these conditions corresponds to the analogous conditions (4.2.16) and (4.2.17) on the affine connection, from which we obtained to solution branches in the previous section. For these two branches, one finds the following spin connections accompanying the diagonal tetrad:

1. For the vector branch (4.2.24), the non-vanishing spin connection components are given by

$$\begin{gathered}
\omega'^1{}_{2\vartheta} = -\omega'^2{}_{1\vartheta} = -\chi\,, \quad \omega'^1{}_{3\varphi} = -\omega'^3{}_{1\varphi} = -\chi\sin\vartheta\,, \quad \omega'^2{}_{3\varphi} = -\omega'^3{}_{2\varphi} = -\cos\vartheta\,,\\
\omega'^0{}_{1r} = \omega'^1{}_{0r} = \mp\frac{iu}{\chi}\,, \quad \omega'^0{}_{2\vartheta} = \omega'^2{}_{0\vartheta} = \mp iur\,, \quad \omega'^0{}_{3\varphi} = \omega'^3{}_{0\varphi} = \mp iur\sin\vartheta\,.
\end{gathered} \tag{4.2.32}$$

2. For the axial branch (4.2.25), one has the non-vanishing spin connection components

$$\begin{gathered}
\omega'^1{}_{3\varphi} = -\omega'^3{}_{1\varphi} = -\chi\sin\vartheta\,, \quad \omega'^2{}_{3r} = -\omega'^3{}_{2r} = \pm\frac{u}{\chi}\,, \quad \omega'^2{}_{3\varphi} = -\omega'^3{}_{2\varphi} = -\cos\vartheta\,,\\
\omega'^1{}_{2\vartheta} = -\omega'^2{}_{1\vartheta} = -\chi\,, \quad \omega'^1{}_{2\varphi} = -\omega'^2{}_{1\varphi} = \pm ur\sin\vartheta\,, \quad \omega'^1{}_{3\vartheta} = -\omega'^3{}_{1\vartheta} = \mp ur\,.
\end{gathered} \tag{4.2.33}$$

One easily checks that these components define a flat, antisymmetric spin connection, corresponding to a metric teleparallel geometry, and so we may denote them with the notation $\overset{\bullet}{\omega}'^A{}_{B\mu}$ introduced earlier. It then follows that there exists a local Lorentz transformation $\Lambda^A{}_B$, which acts on the tetrad and spin connection using the prescription (2.2.28), such that one obtains the Weitzenböck gauge $\overset{\bullet}{\omega}^A{}_{B\mu} \equiv 0$. For the two solution branches, these local Lorentz transformations and resulting tetrads in the Weitzenböck gauge are given as follows.

1. For the vector branch (4.2.32), one may achieve the Weitzenböck gauge using the local Lorentz transformation defined by

$$\Lambda^A{}_B = \begin{pmatrix} \chi & \mp iur\sin\vartheta\cos\varphi & \mp iur\sin\vartheta\sin\varphi & \mp iur\cos\vartheta \\ \mp iur & \chi\sin\vartheta\cos\varphi & \chi\sin\vartheta\sin\varphi & \chi\cos\vartheta \\ 0 & \cos\vartheta\cos\varphi & \cos\vartheta\sin\varphi & -\sin\vartheta \\ 0 & -\sin\varphi & \cos\varphi & 0 \end{pmatrix}. \tag{4.2.34}$$

Applying this local Lorentz transformation to the diagonal tetrad (4.2.30), one obtains the non-diagonal tetrad

$$\theta^0 = \mathcal{N}\chi\mathrm{d}t \pm iu\mathcal{A}\frac{r}{\chi}\mathrm{d}r\,, \tag{4.2.35a}$$
$$\theta^1 = \mathcal{A}\left[\sin\vartheta\cos\varphi\left(\mathrm{d}r \pm iu\frac{\mathcal{N}}{\mathcal{A}}r\mathrm{d}t\right) + r\cos\vartheta\cos\varphi\mathrm{d}\vartheta - r\sin\vartheta\sin\varphi\mathrm{d}\varphi\right], \tag{4.2.35b}$$
$$\theta^2 = \mathcal{A}\left[\sin\vartheta\sin\varphi\left(\mathrm{d}r \pm iu\frac{\mathcal{N}}{\mathcal{A}}r\mathrm{d}t\right) + r\cos\vartheta\sin\varphi\mathrm{d}\vartheta + r\sin\vartheta\cos\varphi\mathrm{d}\varphi\right], \tag{4.2.35c}$$
$$\theta^3 = \mathcal{A}\left[\cos\vartheta\left(\mathrm{d}r \pm iu\frac{\mathcal{N}}{\mathcal{A}}r\mathrm{d}t\right) - r\sin\vartheta\mathrm{d}\vartheta\right]. \tag{4.2.35d}$$

Note that the two sign choices are related by simultaneously performing a (time orientation changing) Lorentz transformation $\theta^0 \mapsto -\theta^0$ and a reparametrization $\mathcal{N}\mathrm{d}t \mapsto -\mathcal{N}\mathrm{d}t$. Hence, both choices describe the same teleparallel geometry.

2. For the axial branch (4.2.33), the Weitzenböck gauge is obtained from the local Lorentz transformation

$$\Lambda^A{}_B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\vartheta\cos\varphi & \sin\vartheta\sin\varphi & \cos\vartheta \\ 0 & \chi\cos\vartheta\cos\varphi \pm ur\sin\varphi & \chi\cos\vartheta\sin\varphi \mp ur\cos\varphi & -\chi\sin\vartheta \\ 0 & \pm ur\cos\vartheta\cos\varphi - \chi\sin\varphi & \chi\cos\varphi \pm ur\cos\vartheta\sin\varphi & \mp ur\sin\vartheta \end{pmatrix}. \tag{4.2.36}$$

The corresponding tetrad in the Weitzenböck gauge, which is obtained by transforming the diagonal tetrad (4.2.30), is then given by

$$\theta^0 = \mathcal{N}\mathrm{d}t\,, \tag{4.2.37a}$$
$$\theta^1 = \mathcal{A}\left[\frac{\sin\vartheta\cos\varphi}{\chi}\mathrm{d}r + r(\chi\cos\vartheta\cos\varphi \pm ur\sin\varphi)\mathrm{d}\vartheta - r\sin\vartheta(\chi\sin\varphi \mp ur\cos\vartheta\cos\varphi)\mathrm{d}\varphi\right], \tag{4.2.37b}$$
$$\theta^2 = \mathcal{A}\left[\frac{\sin\vartheta\sin\varphi}{\chi}\mathrm{d}r + r(\chi\cos\vartheta\sin\varphi \mp ur\cos\varphi)\mathrm{d}\vartheta + r\sin\vartheta(\chi\cos\varphi \pm ur\cos\vartheta\sin\varphi)\mathrm{d}\varphi\right], \tag{4.2.37c}$$
$$\theta^3 = \mathcal{A}\left[\frac{\cos\vartheta}{\chi}\mathrm{d}r - r\chi\sin\vartheta\mathrm{d}\vartheta \mp ur^2\sin^2\vartheta\mathrm{d}\varphi\right]. \tag{4.2.37d}$$

Also in this case one has two sign choices. However, note that these are not related by a local Lorentz transformation and reparametrization, so that they constitute different teleparallel geometries.

We finally remark that the two local Lorentz transformations given above are not unique. Performing in addition a global Lorentz transformation preserves the Weitzenböck spin connection $\overset{\bullet}{\omega}{}^A{}_{B\mu} \equiv 0$, while giving another tetrad. This is regarded as describing the same teleparallel geometry.

4.2.5 Representation theory approach

In [77], we also presented another approach in order to derive the most general cosmologically symmetric teleparallel geometry shown above, which makes use of the notion of symmetry detailed in section 3.2. Since we only consider connected symmetry groups here, generated by the vector fields (4.1.1) for rotations and (4.2.1) for translations, it is sufficient to study the infinitesimal symmetry conditions (3.2.5). To further simplify the procedure, one works in the Weitzenböck gauge, in which a teleparallel geometry which is symmetric under the action of a Lie algebra of generating vector fields is characterized by the existence of a Lie algebra homomorphism $\boldsymbol{\lambda} : \mathfrak{g} \to \mathfrak{so}(1,3)$ from the symmetry algebra to the Lorentz algebra. Hence, to find all symmetric teleparallel geometries for a given symmetry algebra, one must construct all such homomorphisms, and then solve the corresponding field equations. Writing the basis elements of the Lorentz algebra as J_i and K_i, with $i = 1, 2, 3$ and the Lie brackets given by

$$[J_i, J_j] = \epsilon_{ijk} J_k\,, \quad [K_i, K_j] = -\epsilon_{ijk} J_k\,, \quad [K_i, J_j] = \epsilon_{ijk} K_k\,, \tag{4.2.38}$$

these are given as follows:

1. The trivial homomorphism:
$$\boldsymbol{\lambda}(R_i) = \boldsymbol{\lambda}(T_i) = 0\,. \tag{4.2.39}$$

2. The vector homomorphism:
$$\boldsymbol{\lambda}(R_i) = J_i\,, \quad \boldsymbol{\lambda}(T_i) = \pm iu K_i\,. \tag{4.2.40}$$

3. The two-form homomorphisms:
$$\boldsymbol{\lambda}(R_i) = J_i\,, \quad \boldsymbol{\lambda}(T_i) = \pm u J_i\,. \tag{4.2.41}$$

A full derivation is given in our work [H4]. One finds that for the trivial homomorphism no tetrad exists which satisfies the symmetry condition (3.2.5), as we have proven in [85]. For the remaining representations, one can choose an explicit matrix representation of the Lorentz algebra, given by the matrices

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4.2.42a}$$

$$K_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{4.2.42b}$$

It then turns out that the homomorphism (4.2.40) leads to the tetrad (4.2.35), while the homomorphism (4.2.41) leads to the tetrad (4.2.37). Finally, we remark that the two sign choices for the vector case (4.2.40) are related by a basis transformation $K_i \mapsto -K_i$ of the Lorentz algebra, and so constitute conjugate representations. This is not the case for the two-form branch (4.2.41). This finding is closely related to the fact that in the former case, the sign in the tetrad (4.2.35) can be absorbed into a Lorentz transformation and reparametrization, which is not the case for the tetrad (4.2.37).

4.2.6 Cosmological gravity field equations

As an application of the most general cosmologically symmetric teleparallel geometry found above, one may derive the cosmological field equations for a given teleparallel gravity theory. As an instructive example, we display the equations for a generic class of theories defined by the action

$$S_g = \frac{1}{2\kappa^2}\int_M \mathcal{F}(\mathbb{T}_1, \mathbb{T}_2, \mathbb{T}_3)\theta \mathrm{d}^4x\,, \tag{4.2.43}$$

where $\mathcal{F}$ is a free function of the three scalar torsion invariants

$$\mathbb{T}_1 = \overset{\bullet}{T}{}^{\mu\nu\rho}\overset{\bullet}{T}_{\mu\nu\rho}\,, \quad \mathbb{T}_2 = \overset{\bullet}{T}{}^{\mu\nu\rho}\overset{\bullet}{T}_{\rho\nu\mu}\,, \quad \mathbb{T}_3 = \overset{\bullet}{T}{}^{\mu}{}_{\mu\rho}\overset{\bullet}{T}_{\nu}{}^{\nu\rho}\,. \tag{4.2.44}$$

The field equations derived from this action are given by

$$\begin{aligned}\kappa^2\Theta_{\mu\nu} = \kappa^2 E_{\mu\nu} = {}& \frac{1}{2}\mathcal{F}g_{\mu\nu} + 2\overset{\circ}{\nabla}{}^{\rho}\left(\mathcal{F}_{,1}\overset{\bullet}{T}_{\nu\mu\rho} + \mathcal{F}_{,2}\overset{\bullet}{T}_{[\rho\mu]\nu} + \mathcal{F}_{,3}\overset{\bullet}{T}{}^{\sigma}{}_{\sigma[\rho}g_{\mu]\nu}\right)\\ &+ \mathcal{F}_{,1}\overset{\bullet}{T}{}^{\rho\sigma}{}_{\mu}\left(\overset{\bullet}{T}_{\nu\rho\sigma} - 2\overset{\bullet}{T}_{[\rho\sigma]\nu}\right) - \frac{1}{2}\mathcal{F}_{,3}\overset{\bullet}{T}{}^{\sigma}{}_{\sigma\rho}\left(\overset{\bullet}{T}{}^{\rho}{}_{\mu\nu} + 2\overset{\bullet}{T}_{(\mu\nu)}{}^{\rho}\right)\\ &+ \frac{1}{2}\mathcal{F}_{,2}\left[\overset{\bullet}{T}_{\mu}{}^{\rho\sigma}\left(2\overset{\bullet}{T}_{\rho\sigma\nu} - \overset{\bullet}{T}_{\nu\rho\sigma}\right) + \overset{\bullet}{T}{}^{\rho\sigma}{}_{\mu}\left(2\overset{\bullet}{T}_{[\rho\sigma]\nu} - \overset{\bullet}{T}_{\nu\rho\sigma}\right)\right]\,,\end{aligned} \tag{4.2.45}$$

where commas denote derivatives with respect to the numbered arguments. By imposing cosmological symmetry on these field equations, it follows that the energy-momentum tensor must take the perfect fluid form

$$\Theta_{\mu\nu} = (\rho + p)n_\mu n_\nu + pg_{\mu\nu} = \rho n_\mu n_\nu + ph_{\mu\nu}\,, \tag{4.2.46}$$

where the matter density ρ and pressure p are functions of time only. For the gravitational side of the field equations, this implied that they must be of the same form

$$E_{\mu\nu} = (\mathfrak{N} + \mathfrak{H})n_\mu n_\nu + \mathfrak{H}g_{\mu\nu} = \mathfrak{N}n_\mu n_\nu + \mathfrak{H}h_{\mu\nu}\,, \tag{4.2.47}$$

and so the cosmological field equations take the general form

$$\rho = \mathfrak{N}\,, \quad p = \mathfrak{H}\,. \tag{4.2.48}$$

The expressions on the right hand side depend both on the gravity theory under consideration and the chosen cosmologically symmetric tetrad. For the action (4.2.43) and the tetrad (4.2.37) in the Weitzenböck gauge, or equivalently the diagonal tetrad (4.2.30) with the spin connection (4.2.33), one finds the cosmological field equations

$$\kappa^2\rho = -\frac{\mathcal{F}}{2} + 3H^2(2\mathcal{F}_{,1} + \mathcal{F}_{,2} + 3\mathcal{F}_{,3})\,, \tag{4.2.49a}$$

$$\begin{aligned}\kappa^2 p = {}& \frac{\mathcal{F}}{2} - (\dot{H} + 3H^2)(2\mathcal{F}_{,1} + \mathcal{F}_{,2} + 3\mathcal{F}_{,3})\\ &+ 8\frac{u^2}{\mathcal{A}^2}\left[\mathcal{F}_{,1} - \mathcal{F}_{,2} + 6H^2\left(\mathcal{F}_{,22} + 3\mathcal{F}_{,23} - 2\mathcal{F}_{,11} + \mathcal{F}_{,12} - 3\mathcal{F}_{,13}\right)\right]\end{aligned}$$

$$- 6H^2\dot{H}\left(\mathcal{F}_{,11} + 4\mathcal{F}_{12} + 12\mathcal{F}_{,13} + \mathcal{F}_{,22} + 6\mathcal{F}_{,23} + 9\mathcal{F}_{,33}\right) , \tag{4.2.49b}$$

while the diagonal tetrad with the connection (4.2.32), or equivalently the tetrad (4.2.35) in the Weitzenböck gauge yields the field equations

$$\kappa^2\rho = -\frac{\mathcal{F}}{2} + 3H\left(H - \frac{iu}{\mathcal{A}}\right)(2\mathcal{F}_{,1} + \mathcal{F}_{,2} + 3\mathcal{F}_{,3}) , \tag{4.2.50a}$$

$$\begin{aligned}\kappa^2 p = {} & \frac{\mathcal{F}}{2} - \left(\dot{H} + 3H^2 - 3iH\frac{u}{\mathcal{A}} - \frac{u^2}{\mathcal{A}^2}\right)(2\mathcal{F}_{,1} + \mathcal{F}_{,2} + 3\mathcal{F}_{,3}) \\ & - \left(H - i\frac{u}{\mathcal{A}}\right)^2\left(\dot{H} + iH\frac{u}{\mathcal{A}}\right)\left(\mathcal{F}_{,11} + 4\mathcal{F}_{12} + 12\mathcal{F}_{,13} + \mathcal{F}_{,22} + 6\mathcal{F}_{,23} + 9\mathcal{F}_{,33}\right) ,\end{aligned} \tag{4.2.50b}$$

where we have introduced the Hubble parameter and its cosmological time derivative defined by

$$H = \frac{\dot{\mathcal{A}}}{\mathcal{A}} = \frac{\partial_t\mathcal{A}}{\mathcal{A}\mathcal{N}} , \quad \dot{H} = \frac{\partial_t H}{\mathcal{N}} , \tag{4.2.51}$$

and where the function $\mathcal{F}$ and its derivatives must be evaluated at the cosmological values

$$\mathbb{T}_1 = 6(4\mathcal{T}_2^2 - \mathcal{T}_1^2) , \quad \mathbb{T}_2 = -3(8\mathcal{T}_2^2 + \mathcal{T}_1^2) , \quad \mathbb{T}_3 = -9\mathcal{T}_1^2 . \tag{4.2.52}$$

In the limit $u \to 0$, in which both cosmologically symmetric teleparallel geometry branches meet, both field equations take the common form

$$\kappa^2\rho = -\frac{\mathcal{F}}{2} + 3H^2(2\mathcal{F}_{,1} + \mathcal{F}_{,2} + 3\mathcal{F}_{,3}) , \tag{4.2.53a}$$

$$\begin{aligned}\kappa^2 p = {} & \frac{\mathcal{F}}{2} - (\dot{H} + 3H^2)(2\mathcal{F}_{,1} + \mathcal{F}_{,2} + 3\mathcal{F}_{,3}) \\ & - 6H^2\dot{H}\left(\mathcal{F}_{,11} + 4\mathcal{F}_{12} + 12\mathcal{F}_{,13} + \mathcal{F}_{,22} + 6\mathcal{F}_{,23} + 9\mathcal{F}_{,33}\right) .\end{aligned} \tag{4.2.53b}$$

It is remarkable that although the different tetrad branches yield the same Friedmann-Lemaître-Robertson-Walker metric (4.2.2), they exhibit different cosmological dynamics, thus resulting in a different evolution of the cosmological scale factor. However, one may argue that depending on the sign of the curvature parameter $k = u^2$, there is always exactly one real tetrad branch, and so this branch must be chosen in order to obtain the correct dynamics for this sign choice. It is nonetheless remarkable that even following this argument, one obtains essentially different dynamics for both cases, which exceed the simple change of sign which happens in curvature-based gravity theories, where k enters through the metric only.

5 Perturbed spacetime symmetries

In the previous sections we have studied geometries which are invariant under the action $\psi : G \times M \to M$ of a transformation group G on the spacetime manifold M. Already this simple assumption captures various physically motivated examples, including homogeneous and isotropic cosmology and spherically symmetric gravitating objects. Further examples can be investigated by studying perturbations around such exact symmetric spacetime. In this case the symmetry of the background spacetime leaves an imprint on the structure of the perturbation, allowing them to be decomposed into irreducible representations of the symmetry group, thus leading to a significantly simplified description. In the following two sections we discuss two emanations of this concept: gauge-invariant linear cosmological perturbations in teleparallel gravity in section 5.1 and gauge-invariant post-Newtonian perturbations in section 5.2.

5.1 Gauge-invariant linear perturbations in teleparallel cosmology

An important framework in order to test the viability of gravity theories using observations in cosmology, such as the cosmic microwave background, is the theory of cosmological perturbations. Its most convenient formulation makes use of the invariance of a gravity theory under consideration under infinitesimal diffeomorphisms, which allow the description of the perturbed geometry of spacetime in terms of gauge-invariant quantities. Conventionally, this framework has been employed in theories whose dynamical field variable is the metric. It has also been used in the study of teleparallel gravity theories, which employ a tetrad as a fundamental field variable, but this study has been limited to spatially flat backgrounds. In our work [H5] we extended the framework of gauge-invariant cosmological perturbations to all cosmologically symmetric backgrounds presented in section 4.2. Here we provide a summary of this work, and complement the decomposition of the teleparallel geometry modeling the gravitational interaction with a discussion of the corresponding matter components, to obtain a complete view of the formalism. In section 5.1.1, we explain the split of the tetrad perturbation in time and space components. In section 5.1.2, we display their transformation under infinitesimal diffeomorphisms, from which we derive gauge-invariant perturbation variables in section 5.1.3. We then apply this decomposition to the perturbed gravitational field equation, whose structure we discuss in section 5.1.4: we decompose the gravitational side in section 5.1.5 and the energy-momentum side in section 5.1.6. This leads to the fully gauge-invariant field equations in section 5.1.7. We show how these are subject to the Bianchi identities, which are complemented by the energy-momentum conservation of matter, in section 5.1.8. As an illustrative example, we derived the perturbed cosmological field equations of the teleparallel equivalent of general relativity in section 5.1.9.

5.1.1 Space-time split of linear perturbations

In the following, we will discuss linear perturbations of the cosmologically symmetric teleparallel geometries displayed in section 4.2. For simplicity, we work in the Weitzenböck gauge $\overset{\bullet}{\omega}{}^A{}_{B\mu} \equiv 0$, and so the background tetrad $\bar{\theta}^A{}_\mu$ is given by one of the tetrads (4.2.35) and (4.2.37), depending on the choice of the branch to be considered. Imposing the Weitzenböck gauge also for the perturbations, a general perturbation of this geometry is then given by a tetrad perturbation of the form

$$\theta^A{}_\mu = \bar{\theta}^A{}_\mu + \delta\theta^A{}_\mu \,. \tag{5.1.1}$$

While it is entirely possible to directly use this form of the perturbations, which is conventionally done for a diagonal background tetrad [25, 51, 49], it turns out to be rather cumbersome for the non-diagonal background tetrads we consider here. However, this difficulty can be circumvented by introducing the perturbations

$$\tau_{\mu\nu} = \eta_{AB}\bar{\theta}^A{}_\mu \delta\theta^B{}_\nu \,, \tag{5.1.2}$$

which carry only spacetime indices [158]. This approach significantly simplifies the decomposition of the tetrad perturbations into their temporal and spatial parts, and thus the resulting field equations.

In order to perform the aforementioned $3+1$ decomposition into spatial and temporal components, we now assume that the spacetime manifold M is globally hyperbolic, and so has the form $M \cong \mathbb{R} \times \Sigma$, with a purely spatial manifold Σ. We then work in coordinates $(x^\mu) = (t, x^a)$ which respect this product structure, such that t is the time coordinate on $\mathbf{R}$, while (x^a) are the spatial coordinates on Σ. The former carries a canonical metric $\mathrm{d}t \otimes \mathrm{d}t$, while we assume the latter to be a maximally symmetric space, equipped with a metric

$$\gamma_{ab}\mathrm{d}x^a \otimes \mathrm{d}x^b = \frac{\mathrm{d}r \otimes \mathrm{d}r}{1 - u^2r^2} + r^2(\mathrm{d}\vartheta \otimes \mathrm{d}\vartheta + \sin^2\vartheta\, \mathrm{d}\varphi \otimes \mathrm{d}\varphi) \tag{5.1.3}$$

in spherical coordinates, where u is the curvature parameter appearing in the background tetrad. This allows us to write the background metric (4.2.2) on M as a warped product metric

$$\bar{g}_{\mu\nu}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = \eta_{AB}\bar{\theta}^A{}_\mu\bar{\theta}^B{}_\nu\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = -\mathcal{N}^2\mathrm{d}t \otimes \mathrm{d}t + \mathcal{A}^2\gamma_{ab}\mathrm{d}x^a \otimes \mathrm{d}x^b\,, \tag{5.1.4}$$

where we identify the induced metric on the spatial hypersurfaces of constant time t as

$$h_{\mu\nu}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = \mathcal{A}^2\gamma_{ab}\mathrm{d}x^a \otimes \mathrm{d}x^b\,. \tag{5.1.5}$$

Similarly, we write υ_{abc} for the Levi-Civita tensor of γ_{ab}, which is related to the Levi-Civita tensor $\varepsilon_{\mu\nu\rho}$ of $h_{\mu\nu}$ by

$$\varepsilon_{\mu\nu\rho}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu \otimes \mathrm{d}x^\rho = \mathcal{A}^3\upsilon_{abc}\mathrm{d}x^a \otimes \mathrm{d}x^b \otimes \mathrm{d}x^c\,, \tag{5.1.6}$$

as follows from the relation (4.2.7).

In the following we will be working with linear perturbations, and so the background metric will be used for raising and lowering indices. However, since the components of the spacetime metric $\bar{g}_{\mu\nu}$ depend on time through the time-dependence of $\mathcal{N}$ and $\mathcal{A}$, using $\bar{\gamma}_{\mu\nu}$ for this purpose would not commute with taking time derivatives, so that for a vector field X^μ one has

$$\partial_t X_\mu = \partial_t(\bar{g}_{\mu\nu}X^\nu) \neq \bar{g}_{\mu\nu}\partial_t X^\nu\,. \tag{5.1.7}$$

It is therefore more convenient to define the space and time components as tensor fields on Σ, and regard the time t as an extrinsic parameter, so that one can use the metric γ_{ab} in order to raise and lower indices on spatial tensors. To distinguish such spatial tensors on Σ, which now carry a dependence on the extrinsic time parameter t, from tensors on the spacetime manifold Σ, we denote them with a hat. Using the metric γ_{ab} on Σ has the advantage that it does not depend on the time t, and so raising and lowering indices commutes with taking time derivatives. Thus, expressions of the form

$$\dot{\hat{X}}_a = \partial_t\hat{X}_a = \partial_t(\gamma_{ab}\hat{X}^b) = \gamma_{ab}\partial_t\hat{X}^b = \gamma_{ab}\dot{\hat{X}}^b \tag{5.1.8}$$

have a well-defined meaning. In order to connect this convention to the usual raising and lowering of indices for tensor fields on M, suitable factors must be introduced. For a vector field $X = X^\mu\partial_\mu$, we write

$$X = \mathcal{N}^{-1}\hat{X}^0\partial_t + \mathcal{A}^{-1}\hat{X}^a\partial_a\,, \quad \hat{X}^0 = -n_\mu X^\mu = \mathcal{N}X^0\,, \quad \hat{X}^a = \mathcal{A}X^a \tag{5.1.9}$$

for the temporal and spatial components. Conversely, for a covector field $\alpha = \alpha_\mu\mathrm{d}x^\mu$ we write

$$\alpha = \mathcal{N}\hat{\alpha}_0\,\mathrm{d}t + \mathcal{A}\hat{\alpha}_a\,\mathrm{d}x^a\,, \quad \hat{\alpha}_0 = n^\mu\alpha_\mu = \mathcal{N}^{-1}\alpha_0\,, \quad \hat{\alpha}_a = \mathcal{A}^{-1}\alpha_a\,. \tag{5.1.10}$$

Multiple factors are introduced if there are multiple indices present.

Also for the covariant derivatives on M a suitable decomposition must be performed. For this purpose, it is helpful to introduce the Levi-Civita covariant derivative d_a defined by the metric γ_{ab} on Σ, which acts on spatial tensor fields. Given vector fields $X = X^\mu\partial_\mu, Z = Y^\mu\partial_\mu, Z = Z^\mu\partial_\mu$, related by

$$\mathcal{N}^{-1}\hat{Z}^0\partial_t + \mathcal{A}^{-1}\hat{Z}^a\partial_a = Z = \overset{\circ}{\nabla}_X Y = (X^\nu\overset{\circ}{\nabla}_\nu Y^\mu)\partial_\mu\,, \tag{5.1.11}$$

one finds that the space and time components of the covariant derivative are given by

$$\hat{Z}^0 = \mathcal{N}^{-1}\hat{X}^0\partial_t\hat{Y}^0 + \mathcal{A}^{-1}\hat{X}^a\mathrm{d}_a\hat{Y}^0 + H\gamma_{ab}\hat{X}^a\hat{Y}^b\,, \tag{5.1.12a}$$
$$\hat{Z}^a = \mathcal{N}^{-1}\hat{X}^0\partial_t\hat{Y}^a + \mathcal{A}^{-1}\hat{X}^b\mathrm{d}_b\hat{Y}^b + H\hat{X}^a\hat{Y}^0\,, \tag{5.1.12b}$$

where the Hubble parameter (4.2.51) enters through its appearance in the coefficients $\overset{\circ}{\Gamma}{}^\mu{}_{\nu\rho}$ of the Levi-Civita connection.

For the tetrad perturbation (5.1.2) introduced above, the decomposition into spatial and temporal parts gives rise to a scalar $\hat{\tau}_{00}$, two vectors $\hat{\tau}_{a0}$ and $\hat{\tau}_{0b}$, as well es a rank-2 tensor $\hat{\tau}_{ab}$. These spatial tensor fields are further decomposed into the irreducible components

$$\hat{\tau}_{00} = \mathcal{N}^{-2}\tau_{00} = \hat{\phi}\,, \quad \hat{\tau}_{0b} = (\mathcal{A}\mathcal{N})^{-1}\tau_{0b} = \mathrm{d}_b\hat{j} + \hat{b}_b\,, \quad \hat{\tau}_{a0} = (\mathcal{A}\mathcal{N})^{-1}\tau_{a0} = \mathrm{d}_a\hat{y} + \hat{v}_a\,,$$
$$\hat{\tau}_{ab} = \mathcal{A}^{-2}\tau_{ab} = \hat{\psi}\gamma_{ab} + \mathrm{d}_a\mathrm{d}_b\hat{\sigma} + \mathrm{d}_b\hat{c}_a + \upsilon_{abc}(\mathrm{d}^c\hat{\xi} + \hat{w}^c) + \frac{1}{2}\hat{q}_{ab}\,. \tag{5.1.13}$$

Here $\hat{\phi}, \hat{j}, \hat{y}, \hat{\psi}, \hat{\sigma}$ are scalars, $\hat{\xi}$ is a pseudoscalar, $\hat{b}_a, \hat{v}_a, \hat{c}_a$ are divergence-free vectors, $\hat{w}_a$ is a divergence-free pseudovector and $\hat{q}_{ab}$ is a trace-free, divergence-free, symmetric tensor. They are subject to the conditions

$$\mathrm{d}_a\hat{b}^a = \mathrm{d}_a\hat{v}^a = \mathrm{d}_a\hat{c}^a = \mathrm{d}_a\hat{w}^a = 0\,, \quad \mathrm{d}_a\hat{q}^{ab} = 0\,, \quad \hat{q}_{[ab]} = 0\,, \quad \hat{q}_a{}^a = 0\,, \tag{5.1.14}$$

where d_a denotes the Levi-Civita covariant derivative of the maximally symmetric spatial background metric γ_{ab}. In the following sections, we will discuss how to use these components in order to express the perturbed cosmological field equations in teleparallel gravity.

5.1.2 Gauge transformation

One of the advantages of the decomposition (5.1.13) becomes apparent if we consider gauge transformations, i.e., changes of the perturbations under an infinitesimal coordinate transformation. Such a transformation is equivalent to an infinitesimal diffeomorphism, generated by a vector field X^μ, which changes the coordinates to

$$x'^\mu = x^\mu + X^\mu + \mathcal{O}([X]^2)\,. \tag{5.1.15}$$

Under this transformation, the components of tensor fields change by their Lie derivatives. In particular, for the tetrad one thus finds the relation

$$\theta^A{}_\mu = \theta'^A{}_\mu + (\pounds_X\theta')^A{}_\mu + \mathcal{O}([X]^2) = \theta'^A{}_\mu + (\pounds_X\bar{\theta})^A{}_\mu + \mathcal{O}([X, \delta\theta]^2)\,, \tag{5.1.16}$$

where the last expression denotes the fact that we neglect any terms which are of more than linear order in the vector field X^μ, the perturbation $\delta\theta^A{}_\mu$ or their product, hence allowing us to replace the Lie derivative of $\theta^A{}_\mu$ by that of the background tetrad $\bar{\theta}^A{}_\mu$, which is already of linear order in X^μ. In then follows that also $\theta'^A{}_\mu$ can be regarded as a perturbation around the same background $\bar{\theta}^A{}_\mu$, where the perturbations are related by

$$\delta\theta^A{}_\mu - \delta\theta'^A{}_\mu = (\pounds_X\bar{\theta})^A{}_\mu\,, \tag{5.1.17}$$

where we have no omitted any higher order terms, and kept only the linear perturbation order. Further rewriting the transformed perturbation $\delta\theta'^A{}_\mu$ to have two lower spacetime indices, thus defining $\tau'_{\mu\nu}$ in analogy to $\tau_{\mu\nu}$, one finds

$$\delta_X\tau_{\mu\nu} = \tau_{\mu\nu} - \tau'_{\mu\nu} = \overset{\bullet}{\bar{\nabla}}_\nu X_\mu - \bar{T}_{\mu\nu}{}^\rho X_\rho = \overset{\circ}{\bar{\nabla}}_\nu X_\mu + \bar{K}_{\mu\nu}{}^\rho X_\rho\,. \tag{5.1.18}$$

The transformation $\delta_X\tau_{\mu\nu}$ of the tetrad perturbation can further be decomposed into the irreducible components (5.1.13) we introduced earlier. To achieve this decomposition, one decomposes the vector field X^μ in the form

$$\hat{X}_0 = \hat{X}_\perp\,, \quad \hat{X}_a = \mathrm{d}_a\hat{X}_\| + \hat{Z}_a \tag{5.1.19}$$

into two scalars $\hat{X}_\perp$ and $\hat{X}_\|$ as well as a divergence-free vector $\hat{Z}_a$, which satisfies $\mathrm{d}_a\hat{Z}^a = 0$. Decomposing the gauge transformation (5.1.18), one finds that the irreducible components obey the transformation rule

$$\begin{gathered}\delta_X\hat{\psi} = -\frac{\hat{X}_\perp\partial_t\mathcal{A}}{\mathcal{N}\mathcal{A}}\,, \quad \delta_X\hat{\sigma} = \frac{\hat{X}_\|}{\mathcal{A}}\,, \quad \delta_X\hat{y} = \frac{\partial_t\hat{X}_\|}{\mathcal{N}} - \frac{\mathfrak{v}\hat{X}_\|}{\mathcal{A}}\,,\\ \delta_X\hat{j} = \frac{\mathcal{N}\hat{X}_\perp + (\mathcal{N}\mathfrak{v} - \partial_t\mathcal{A})\hat{X}_\|}{\mathcal{N}\mathcal{A}}\,, \quad \delta_X\hat{\xi} = -\frac{\mathfrak{a}\hat{X}_\|}{\mathcal{A}}\,, \quad \delta_X\hat{\phi} = \frac{\partial_t\hat{X}_\perp}{\mathcal{N}}\end{gathered} \tag{5.1.20}$$

for the (pseudo-)scalars,

$$\delta_X\hat{c}_a = \frac{\hat{Z}_a}{\mathcal{A}}\,, \quad \delta_X\hat{v}_a = \frac{\partial_t\hat{Z}_a}{\mathcal{N}} - \frac{\mathfrak{v}\hat{Z}_a}{\mathcal{A}}\,, \quad \delta_X\hat{b}_a = \frac{(\mathcal{N}\mathfrak{v} - \partial_t\mathcal{A})\hat{Z}_a}{\mathcal{N}\mathcal{A}}\,, \quad \delta_X\hat{w}_a = -\frac{\mathfrak{a}\hat{Z}_a}{\mathcal{A}} \tag{5.1.21}$$

for the (pseudo-)vectors, as well as

$$\delta_X\hat{q}_{ab} = 0 \tag{5.1.22}$$

for the symmetric, trace-free tensor component. Note in particular that the right hand side contains the torsion components $\mathfrak{a}$ and $\mathfrak{v}$ which are different for the two branches of cosmologically symmetric teleparallel geometries. This is related to the fact that the background torsion enters into the gauge transformation (5.1.18).

5.1.3 Gauge-invariant perturbations

From the gauge transformation found in the previous section it is now straightforward to construct gauge-invariant combinations of the irreducible perturbation components. For this purpose, one first realizes that the components (5.1.19) of the generating vector field can be expressed in the form

$$\hat{X}_\| = \mathcal{A}\delta_X\hat{\sigma}\,, \quad \hat{X}_\perp = \mathcal{A}\left[\delta_X\hat{j} + \left(\frac{\partial_t\mathcal{A}}{\mathcal{N}} - \mathfrak{v}\right)\delta_X\hat{\sigma}\right]\,, \quad \hat{Z}_a = \mathcal{A}\delta_X\hat{c}_a \tag{5.1.23}$$

in terms of the resulting transformation of the perturbation components $\hat{\sigma}, \hat{j}$ and $\hat{c}_a$. For each of the remaining perturbation components, one can then eliminate the corresponding induced transformation components. This yields the gauge-invariant (pseudo-)scalars

$$\begin{gathered}\hat{\boldsymbol{\xi}} = \hat{\xi} + \mathfrak{a}\hat{\sigma}\,, \quad \hat{\mathbf{y}} = \hat{y} - \frac{\mathcal{A}\partial_t\hat{\sigma}}{\mathcal{N}} - \left(\frac{\partial_t\mathcal{A}}{\mathcal{N}} - \mathfrak{v}\right)\hat{\sigma}\,, \quad \hat{\boldsymbol{\psi}} = \hat{\psi} + \frac{\partial_t\mathcal{A}}{\mathcal{N}}\left[\hat{j} + \left(\frac{\partial_t\mathcal{A}}{\mathcal{N}} - \mathfrak{v}\right)\hat{\sigma}\right]\,,\\ \hat{\boldsymbol{\phi}} = \hat{\phi} - \frac{\partial_t\mathcal{A}}{\mathcal{N}}\left[\hat{j} + \left(\frac{\partial_t\mathcal{A}}{\mathcal{N}} - \mathfrak{v}\right)\hat{\sigma}\right] - \frac{\mathcal{A}}{\mathcal{N}}\partial_t\left[\hat{j} + \left(\frac{\partial_t\mathcal{A}}{\mathcal{N}} - \mathfrak{v}\right)\hat{\sigma}\right]\,,\end{gathered} \tag{5.1.24}$$

the gauge-invariant (pseudo-)vectors

$$\hat{\mathbf{v}}_a = \hat{v}_a + \left(\mathfrak{v} - \frac{\partial_t\mathcal{A}}{\mathcal{N}}\right)\hat{c}_a - \frac{\mathcal{A}}{\mathcal{N}}\partial_t\hat{c}_a\,, \quad \hat{\mathbf{b}}_a = \hat{b}_a + \left(\frac{\partial_t\mathcal{A}}{\mathcal{N}} - \mathfrak{v}\right)\hat{c}_a\,, \quad \hat{\mathbf{w}}_a = \hat{w}_a + \mathfrak{a}\hat{c}_a\,, \tag{5.1.25}$$

and the symmetric, trace-free tensor

$$\hat{\mathbf{q}}_{ab} = \hat{q}_{ab}\,, \tag{5.1.26}$$

which is already gauge-invariant by itself. By comparison with the gauge transformation of the irreducible components, one finds that their transformations cancel for the terms given here, so that they are indeed invariant under gauge transformations. In other words, we identify the gauge-invariant tetrad perturbations with the tetrad perturbations in a particular, fixed gauge, which is given by the gauge conditions $\hat{j} = \hat{\sigma} = 0$ and $\hat{c}_a = 0$.

5.1.4 Field equations

In order to derive the dynamics of the gauge-invariant perturbation components defined in the preceding section, we need to consider the gravitational field equations of the gravity theory under consideration. Here we do not specify a particular theory, but consider a generic tetrad field equation of the form

$$\bar{E}_A{}^\mu + \mathfrak{E}_A{}^\mu = E_A{}^\mu = \Theta_A{}^\mu = \bar{\Theta}_A{}^\mu + \mathfrak{T}_A{}^\mu\,, \tag{5.1.27}$$

where the background equations $\bar{E}_A{}^\mu = \bar{\Theta}_A{}^\mu$ satisfy the cosmological symmetry, so that they take the general form (4.2.46) and (4.2.47). Here we have already decomposed the field equations by defining the linear perturbations $\mathfrak{E}_A{}^\mu$ and $\mathfrak{T}_A{}^\mu$. Note that we defined them using mixed indices; since they are linear perturbations, we use the background geometry to transform them into the spacetime expressions

$$\mathfrak{E}_{\mu\nu} = \bar{\theta}^A{}_\mu\bar{g}_{\nu\rho}\mathfrak{E}_A{}^\rho\,, \quad \mathfrak{T}_{\mu\nu} = \bar{\theta}^A{}_\mu\bar{g}_{\nu\rho}\mathfrak{T}_A{}^\rho\,. \tag{5.1.28}$$

In order to express these field equations in terms of the gauge-invariant variables obtained from an irreducible decomposition of the tetrad perturbation, one must apply a similar irreducible decomposition. Here we follow the same approach as in section 5.1.1, where we introduced the 3 + 1 decomposition of tensor fields on the cosmologically symmetric background spacetime. This is shown separately for the gravitational part and the energy-momentum part of the field equations below.

5.1.5 Gravitational field tensor

Following its definition (5.1.28) in the previous section, we now apply an irreducible decomposition to the perturbation $\mathfrak{E}_{\mu\nu}$ of the gravitational part of the field equations. We proceed in full analogy to the decomposition (5.1.13) of the tetrad perturbation, using the same symbols for the irreducible components, but with uppercase letters. Hence, we define the components as

$$\hat{\mathfrak{E}}_{00} = \mathcal{N}^{-2}\mathfrak{E}_{00} = \hat{\Phi}\,, \quad \hat{\mathfrak{E}}_{0b} = (\mathcal{A}\mathcal{N})^{-1}\mathfrak{E}_{0b} = \mathrm{d}_b\hat{J} + \hat{B}_b\,, \quad \hat{\mathfrak{E}}_{a0} = (\mathcal{A}\mathcal{N})^{-1}\mathfrak{E}_{a0} = \mathrm{d}_a\hat{Y} + \hat{V}_a\,,$$
$$\hat{\mathfrak{E}}_{ab} = \mathcal{A}^{-2}\mathfrak{E}_{ab} = \hat{\Psi}\gamma_{ab} + \mathrm{d}_a\mathrm{d}_b\hat{\Sigma} + \mathrm{d}_a\hat{C}_b + \upsilon_{abc}(\mathrm{d}^c\hat{\Xi} + \hat{W}^c) + \frac{1}{2}\hat{Q}_{ab}\,. \tag{5.1.29}$$

Here, in analogy to the irreducible components of $\tau_{\mu\nu}$, the expressions $\hat{\Phi}, \hat{J}, \hat{Y}, \hat{\Psi}, \hat{\Sigma}$ are scalars, $\hat{\Xi}$ is a pseudoscalar, $\hat{B}_a, \hat{V}_a, \hat{C}_a$ are divergence-free vectors, $\hat{W}_a$ is a divergence-free pseudovector and $\hat{Q}_{ab}$ is a trace-free, divergence-free, symmetric tensor. Hence, they are subject to the conditions

$$\mathrm{d}_a\hat{B}^a = \mathrm{d}_a\hat{V}^a = \mathrm{d}_a\hat{C}^a = \mathrm{d}_a\hat{W}^a = 0\,, \quad \mathrm{d}_a\hat{Q}^{ab} = 0\,, \quad \hat{Q}_{[ab]} = 0\,, \quad \hat{Q}_a{}^a = 0\,, \tag{5.1.30}$$

Also note in particular that the term involving $\hat{C}_a$ in the decomposition (5.1.29) has the opposite order of indices compared to the similar term involving $\hat{c}_a$ in the analogous decomposition (5.1.13). The reason for this choice becomes clear when we study gauge transformations of the gravitational field equations. Here we follow the same principle as for the tetrad perturbations shown in section 5.1.2, and use the field equations in the form (5.1.27) as the starting point. Under an infinitesimal diffeomorphism generated by a vector field X^μ the perturbation of the gravitational side of the field equations undergoes the change

$$\delta_X\mathfrak{E}_A{}^\mu = \mathfrak{E}_A{}^\mu - \mathfrak{E}'_A{}^\mu = (\mathcal{L}_X\bar{E})_A{}^\mu = X^\nu\partial_\nu\bar{E}_A{}^\mu - \partial_\nu X^\mu\bar{E}_A{}^\nu\,. \tag{5.1.31}$$

In order to obtain the transformation of the expressions $\mathfrak{E}_{\mu\nu}$ in spacetime indices, we transform the indices with the background geometry, which allows us to to write the gauge transformation in the form

$$\delta_X\mathfrak{E}_{\mu\nu} = \bar{\theta}^A{}_\mu\bar{g}_{\nu\rho}\delta_X\mathfrak{E}_A{}^\rho\,. \tag{5.1.32}$$

Using the form (5.1.19) of the generating vector field, one can decompose this transformation into its time and space components, which yields

$$\delta_X\hat{\mathfrak{E}}_{00} = \frac{\mathfrak{N}\partial_t\hat{X}_\perp - \hat{X}_\perp\partial_t\mathfrak{N}}{\mathcal{N}}\,, \tag{5.1.33a}$$

$$\delta_X\hat{\mathfrak{E}}_{0b} = \frac{[\mathcal{N}\mathfrak{v}\mathfrak{H} - (\mathfrak{N}+\mathfrak{H})\partial_t\mathcal{A}](\mathrm{d}_b\hat{X}_\parallel + \hat{Z}_b) + \mathfrak{N}\mathcal{A}\partial_t(\mathrm{d}_b\hat{X}_\parallel + \hat{Z}_b)}{\mathcal{N}\mathcal{A}}\,, \tag{5.1.33b}$$

$$\delta_X\hat{\mathfrak{E}}_{a0} = \frac{(\mathcal{N}\mathfrak{v} - \partial_t\mathcal{A})\mathfrak{N}(\mathrm{d}_a\hat{X}_\parallel + \hat{Z}_a) - \mathcal{N}\mathfrak{H}\mathrm{d}_a\hat{X}_\perp}{\mathcal{N}\mathcal{A}}\,, \tag{5.1.33c}$$

$$\delta_X\hat{\mathfrak{E}}_{ab} = \frac{(\mathfrak{H}\partial_t\mathcal{A} - \mathcal{A}\partial_t\mathfrak{H})\hat{X}_\perp\gamma_{ab} - \mathcal{N}\mathfrak{H}\mathfrak{a}\upsilon_{abc}(\mathrm{d}^c\hat{X}_\parallel + \hat{Z}^c) - \mathcal{N}\mathfrak{H}\mathrm{d}_a(\mathrm{d}_b\hat{X}_\parallel + \hat{Z}_b)}{\mathcal{N}\mathcal{A}}\,. \tag{5.1.33d}$$

Further using the decomposition (5.1.29) of the field equations, we can now easily identify the transformation of the irreducible components. We find that the scalar components transform as

$$\delta_X\hat{J} = \frac{[\mathcal{N}\mathfrak{v}\mathfrak{H} - (\mathfrak{N}+\mathfrak{H})\partial_t\mathcal{A}]\hat{X}_\parallel + \mathfrak{N}\mathcal{A}\partial_t\hat{X}_\parallel}{\mathcal{N}\mathcal{A}}\,, \quad \delta_X\hat{\Psi} = \frac{\mathfrak{H}\partial_t\mathcal{A} - \mathcal{A}\partial_t\mathfrak{H}}{\mathcal{N}\mathcal{A}}\hat{X}_\perp\,, \quad \delta_X\hat{\Sigma} = -\frac{\mathfrak{H}\hat{X}_\parallel}{\mathcal{A}}\,,$$
$$\delta_X\hat{\Phi} = \frac{\mathfrak{N}\partial_t\hat{X}_\perp - \hat{X}_\perp\partial_t\mathfrak{N}}{\mathcal{N}}\,, \quad \delta_X\hat{\Xi} = -\frac{\mathfrak{H}\mathfrak{a}\hat{X}_\parallel}{\mathcal{A}}\,, \quad \delta_X\hat{Y} = \frac{(\mathcal{N}\mathfrak{v} - \partial_t\mathcal{A})\mathfrak{N}\hat{X}_\parallel - \mathcal{N}\mathfrak{H}\hat{X}_\perp}{\mathcal{N}\mathcal{A}}\,, \tag{5.1.34}$$

while for the vector components we have

$$\delta_X\hat{V}_a = \frac{(\mathcal{N}\mathfrak{v} - \partial_t\mathcal{A})\mathfrak{N}\hat{Z}_a}{\mathcal{N}\mathcal{A}}\,, \quad \delta_X\hat{C}_a = -\frac{\mathfrak{H}\hat{Z}_a}{\mathcal{A}}\,, \quad \delta_X\hat{W}_a = -\frac{\mathfrak{H}\mathfrak{a}\hat{Z}_a}{\mathcal{A}}\,,$$
$$\delta_X\hat{B}_a = \frac{[\mathcal{N}\mathfrak{v}\mathfrak{H} - (\mathfrak{N}+\mathfrak{H})\partial_t\mathcal{A}]\hat{Z}_a + \mathfrak{N}\mathcal{A}\partial_t\hat{Z}_a}{\mathcal{N}\mathcal{A}}\,, \tag{5.1.35}$$

and finally the tensor component is gauge-invariant,

$$\delta_X\hat{Q}_{ab} = 0\,. \tag{5.1.36}$$

This allows us to proceed in analogy to the tensor component of the tetrad perturbation, and to construct gauge-invariant quantities by adding suitable multiples of the tetrad components $\hat{j}, \hat{\sigma}, \hat{c}_a$, whose transformations then cancel those intrinsic to the components of the field equations. One thus defines the scalars

$$\hat{\mathbf{\Psi}} = \hat{\Psi} - \frac{\mathfrak{H}\partial_t\mathcal{A} - \mathcal{A}\partial_t\mathfrak{H}}{\mathcal{N}}\left[\hat{j} + \left(\frac{\partial_t\mathcal{A}}{\mathcal{N}} - \mathfrak{v}\right)\hat{\sigma}\right], \quad \hat{\mathbf{\Sigma}} = \hat{\Sigma} + \mathfrak{H}\hat{\sigma}, \quad \hat{\mathbf{\Xi}} = \hat{\Xi} + \mathfrak{a}\mathfrak{H}\hat{\sigma},$$
$$\hat{\mathbf{J}} = \hat{J} - \frac{(\mathcal{N}\mathfrak{v} - \partial_t\mathcal{A})\mathfrak{H}\hat{\sigma} + \mathfrak{N}\mathcal{A}\partial_t\hat{\sigma}}{\mathcal{N}}, \quad \hat{\mathbf{Y}} = \hat{Y} + \left(\frac{\partial_t\mathcal{A}}{\mathcal{N}} - \mathfrak{v}\right)(\mathfrak{N} + \mathfrak{H})\hat{\sigma} + \mathfrak{H}\hat{j},$$
$$\hat{\mathbf{\Phi}} = \hat{\Phi} - \frac{\mathfrak{N}\partial_t\mathcal{A} - \mathcal{A}\partial_t\mathfrak{N}}{\mathcal{N}}\left[\hat{j} + \left(\frac{\partial_t\mathcal{A}}{\mathcal{N}} - \mathfrak{v}\right)\hat{\sigma}\right] - \frac{\mathfrak{N}\mathcal{A}}{\mathcal{N}}\partial_t\left[\hat{j} + \left(\frac{\partial_t\mathcal{A}}{\mathcal{N}} - \mathfrak{v}\right)\hat{\sigma}\right], \tag{5.1.37}$$

the vectors

$$\hat{\mathbf{V}}_a = \hat{V}_a + \left(\frac{\partial_t\mathcal{A}}{\mathcal{N}} - \mathfrak{v}\right)\mathfrak{N}\hat{c}_a, \quad \hat{\mathbf{C}}_a = \hat{C}_a + \mathfrak{H}\hat{c}_a,$$
$$\hat{\mathbf{B}}_a = \hat{B}_a - \frac{(\mathcal{N}\mathfrak{v} - \partial_t\mathcal{A})\mathfrak{H}\hat{c}_a + \mathfrak{N}\mathcal{A}\partial_t\hat{c}_a}{\mathcal{N}}, \quad \hat{\mathbf{W}}_a = \hat{W}_a + \mathfrak{a}\mathfrak{H}\hat{c}_a, \tag{5.1.38}$$

as well as the tensor

$$\hat{\mathbf{Q}}_{ab} = \hat{Q}_{ab}, \tag{5.1.39}$$

and finds that these are indeed invariant under gauge transformations. Hence, it follows that for any diffeomorphism-invariant theory of gravity they can be fully expressed in terms of the gauge-invariant tetrad perturbations defined in section 5.1.3. Finally, it is helpful to remark that the gauge-invariant quantities here form the components of a tensor $\boldsymbol{\mathfrak{E}}_{\mu\nu}$, in full analogy to the decomposition (5.1.29). Note that this tensor agrees with the original tensor $\mathfrak{E}_{\mu\nu}$ if and only if the gauge is chosen such that $\hat{j}, \hat{\sigma}, \hat{c}_a$ vanish.

5.1.6 Energy-momentum tensor

The gravitational part of the perturbed gravitational field equations (5.1.27), which we discussed in the previous section, must be complemented by a corresponding perturbation of the energy-momentum tensor, which we discuss next. Recalling that the background energy-momentum tensor $\bar{\Theta}_{\mu\nu}$ must take the perfect fluid form (4.2.46) in order to be compatible with the cosmological symmetry, one finds that by adding a perturbation it takes the general form

$$\Theta_{00} = \mathcal{N}^2(\bar{\rho} + \delta\hat{\rho} - 2\bar{\rho}\hat{\tau}_{00}), \tag{5.1.40a}$$
$$\Theta_{0a} = -\mathcal{A}\mathcal{N}\left[2\bar{\rho}\hat{\tau}_{(0a)} + (\bar{\rho} + \bar{p})\delta\hat{u}_a\right], \tag{5.1.40b}$$
$$\Theta_{ab} = \mathcal{A}^2\left[\bar{p}\gamma_{ab} + 2\bar{p}\hat{\tau}_{(ab)} + \delta\hat{p}\gamma_{ab} + \hat{\pi}_{ab}\right]. \tag{5.1.40c}$$

Here $\bar{\rho}$ and $\bar{p}$ are the background values of the energy density and the pressure, while $\delta\rho = \delta\hat{\rho}$ and $\delta p = \delta\hat{p}$ are their perturbations. The latter are scalars, and therefore do not change when they are pulled back to the spatial slices as argued in section 5.1.1. This is different for the remaining two perturbation components, which are the spatial fluid velocity perturbation $\delta u_a = \mathcal{A}\delta\hat{u}_a$ and the anisotropic stress $\pi_{ab} = \mathcal{A}^2\hat{\pi}_{ab}$. The former has three independent components, while the latter is assumed to be trace-free, $\gamma^{ab}\hat{\pi}_{ab} = 0$, and symmetric, $\hat{\pi}_{[ab]} = 0$, and so has only five independent components. Together with the density and pressure perturbations one thus has ten components, which is to be expected for the energy-momentum tensor. Note that $\delta\hat{u}_a$ and $\hat{\pi}_{ab}$ can further be decomposed into irreducible components, in order to decompose the matter side of the field equations in the same form as the gravitational side. For this purpose, one needs the energy-momentum tensor perturbation (5.1.28) with lower spacetime indices, which follows from its definition as

$$\mathfrak{T}_{\mu\nu} = \Theta_{\mu\nu} - \bar{\Theta}_{\mu\nu} - \bar{g}^{\rho\sigma}(\tau_{\rho\mu}\bar{\Theta}_{\sigma\nu} + 2\tau_{(\nu\rho)}\bar{\Theta}_{\sigma\mu})\,. \tag{5.1.41}$$

Inserting the energy-momentum tensor (5.1.40), and using the decomposition (5.1.13) of the tetrad perturbation, one finds the energy-momentum perturbation

$$\hat{\mathfrak{T}}_{00} = \delta\hat{\rho} + \bar{\rho}\hat{\phi}\,, \tag{5.1.42a}$$
$$\hat{\mathfrak{T}}_{0b} = -\left[(\bar{\rho}+\bar{p})\delta\hat{u}_b + \bar{p}(\hat{v}_b + \mathrm{d}_b\hat{y})\right]\,, \tag{5.1.42b}$$
$$\hat{\mathfrak{T}}_{a0} = -\left[(\bar{\rho}+\bar{p})(\delta\hat{u}_a + \hat{v}_a + \mathrm{d}_a\hat{y}) + \bar{p}(\hat{b}_a + \mathrm{d}_a\hat{j})\right]\,, \tag{5.1.42c}$$
$$\hat{\mathfrak{T}}_{ab} = \delta\hat{p}\gamma_{ab} + \hat{\pi}_{ab} - \bar{p}\left[\hat{\psi}\gamma_{ab} + \mathrm{d}_b\mathrm{d}_a\hat{\sigma} + \mathrm{d}_a\hat{c}_b - \upsilon_{abc}(\mathrm{d}^c\hat{\xi} + \hat{w}^c) + \frac{1}{2}\hat{q}_{ab}\right]\,. \tag{5.1.42d}$$

In order to work in a fully gauge-invariant formalism, also the matter variables must be expressed through gauge-invariant quantities, which are separated from the pure gauge quantities. In order to obtain these quantities, one studies the gauge transformation of the energy-momentum tensor perturbation, which is defined in full analogy to that of the gravitational side of the field equations shown in the previous section. Its split into time and space components therefore has the same form as the corresponding split (5.1.33), with $\bar{\rho}$ in place of $\mathfrak{N}$ and $\bar{p}$ in place of $\mathfrak{H}$, and thus reads

$$\delta_X\hat{\mathfrak{T}}_{00} = \frac{\bar{\rho}\partial_t\hat{X}_\perp - \hat{X}_\perp\partial_t\bar{\rho}}{\mathcal{N}}\,, \tag{5.1.43a}$$
$$\delta_X\hat{\mathfrak{T}}_{0b} = \frac{[\mathcal{N}\mathfrak{v}\bar{p} - (\bar{\rho}+\bar{p})\partial_t\mathcal{A}](\mathrm{d}_b\hat{X}_\parallel + \hat{Z}_b) + \bar{\rho}\mathcal{A}\partial_t(\mathrm{d}_b\hat{X}_\parallel + \hat{Z}_b)}{\mathcal{N}\mathcal{A}}\,, \tag{5.1.43b}$$
$$\delta_X\hat{\mathfrak{T}}_{a0} = \frac{(\mathcal{N}\mathfrak{v} - \partial_t\mathcal{A})\bar{p}(\mathrm{d}_a\hat{X}_\parallel + \hat{Z}_a) - \mathcal{N}\bar{p}\mathrm{d}_a\hat{X}_\perp}{\mathcal{N}\mathcal{A}}\,, \tag{5.1.43c}$$
$$\delta_X\hat{\mathfrak{T}}_{ab} = \frac{(\bar{p}\partial_t\mathcal{A} - \mathcal{A}\partial_t\bar{p})\hat{X}_\perp\gamma_{ab} - \mathcal{N}\bar{p}\mathfrak{a}\upsilon_{abc}(\mathrm{d}^c\hat{X}_\parallel + \hat{Z}^c) - \mathcal{N}\bar{p}\mathrm{d}_a(\mathrm{d}_b\hat{X}_\parallel + \hat{Z}_b)}{\mathcal{N}\mathcal{A}}\,. \tag{5.1.43d}$$

By comparison with the components (5.1.42) of the energy-momentum tensor perturbation, one thus finds that the matter variables obey the transformation rules

$$\delta_X\delta\hat{\rho} = -\frac{\hat{X}_\perp\partial_t\bar{\rho}}{\mathcal{N}}\,, \quad \delta_X\delta\hat{p} = -\frac{\hat{X}_\perp\partial_t\bar{p}}{\mathcal{N}}\,, \quad \delta_X\delta\hat{u}_a = -\frac{\mathcal{A}}{\mathcal{N}}\partial_t\frac{\hat{Z}_a + \mathrm{d}_a\hat{X}_\parallel}{\mathcal{A}}\,, \quad \delta_X\hat{\pi}_{ab} = 0\,. \tag{5.1.44}$$

It is then straightforward to define gauge-invariant matter variables, following again the same procedure as for the tetrad perturbations and the gravitational side of the field equations. From the transformations above one reads off the gauge-invariant energy density and pressure given by

$$\hat{\mathcal{E}} = \delta\hat{\rho} + \frac{\mathcal{A}\partial_t\bar{\rho}}{\mathcal{N}}\left[\hat{j} + \left(\frac{\partial_t\mathcal{A}}{\mathcal{N}} - \mathfrak{v}\right)\hat{\sigma}\right]\,, \quad \hat{\mathcal{P}} = \delta\hat{p} + \frac{\mathcal{A}\partial_t\bar{p}}{\mathcal{N}}\left[\hat{j} + \left(\frac{\partial_t\mathcal{A}}{\mathcal{N}} - \mathfrak{v}\right)\hat{\sigma}\right]\,. \tag{5.1.45}$$

For the velocity perturbation $\delta\hat{u}_a$, it is helpful to first perform a decomposition into a longitudinal scalar component and a transverse divergence-free vector component. These give rise to two gauge-invariant variables via

$$\hat{\mathcal{X}}_a + \mathrm{d}_a\hat{\mathcal{L}} = \delta\hat{u}_a + \frac{\mathcal{A}}{\mathcal{N}}\partial_t(\hat{c}_a + \mathrm{d}_a\hat{\sigma})\,. \tag{5.1.46}$$

Finally, $\hat{\pi}_{ab}$ is already gauge-invariant. It can further be decomposed into a scalar, vector and tensor component,

$$\hat{\pi}_{ab} = \mathrm{d}_a\mathrm{d}_b\hat{\mathcal{S}} - \frac{1}{3}\triangle\hat{\mathcal{S}}\gamma_{ab} + \mathrm{d}_{(a}\hat{\mathcal{V}}_{b)} + \hat{\mathcal{T}}_{ab}\,, \tag{5.1.47}$$

each of which is gauge-invariant on its own. Here we used the spatial Laplace operator $\triangle = \mathrm{d}_a\mathrm{d}^a$. In analogy to the gauge-invariant tensor $\mathfrak{E}_{\mu\nu}$ defined in the previous section, one can then define the corresponding gauge-invariant energy-momentum tensor perturbation as

$$\hat{\mathfrak{T}}_{00} = \hat{\mathcal{E}} + \bar{\rho}\hat{\phi}\,, \tag{5.1.48a}$$

$$\hat{\mathfrak{T}}_{0b} = -\left[(\bar{\rho}+\bar{p})(\hat{\mathcal{X}}_b + \mathrm{d}_b\hat{\mathcal{L}}) + \bar{p}(\hat{\mathbf{v}}_b + \mathrm{d}_b\hat{\mathbf{y}})\right]\,, \tag{5.1.48b}$$

$$\hat{\mathfrak{T}}_{a0} = -\left[(\bar{\rho}+\bar{p})(\hat{\mathcal{X}}_a + \mathrm{d}_a\hat{\mathcal{L}} + \hat{\mathbf{v}}_a + \mathrm{d}_a\hat{\mathbf{y}}) + \bar{p}\hat{\mathbf{b}}_a\right]\,, \tag{5.1.48c}$$

$$\hat{\mathfrak{T}}_{ab} = \left(\hat{\mathcal{P}} - \frac{1}{3}\mathrm{d}_c\mathrm{d}^c\hat{\mathcal{S}} - \bar{p}\hat{\psi}\right)\gamma_{ab} + \mathrm{d}_a\mathrm{d}_b\hat{\mathcal{S}} + \mathrm{d}_{(a}\hat{\mathcal{V}}_{b)} + \hat{\mathcal{T}}_{ab} + \bar{p}\left[\upsilon_{abc}(\mathrm{d}^c\hat{\boldsymbol{\xi}} + \hat{\mathbf{w}}^c) - \frac{1}{2}\hat{\mathbf{q}}_{ab}\right]\,. \tag{5.1.48d}$$

It is evident that these two tensors constitute the gauge-invariant perturbed field equations.

5.1.7 Gauge-invariant field equations

It is convenient to decompose the gauge-invariant perturbed field equations

$$\mathfrak{E}_{\mu\nu} = \mathfrak{T}_{\mu\nu} \tag{5.1.49}$$

into their irreducible components. For the gravitational side, these are the components we constructed in section (5.1.4), while for the energy-momentum tensor they can be obtained from the decomposition (5.1.48). The irreducible components are the six scalar equations

$$\hat{\mathbf{J}} = -(\bar{\rho}+\bar{p})\hat{\mathcal{L}} - \bar{p}\hat{\mathbf{y}}\,,\quad \hat{\boldsymbol{\Sigma}} = \hat{\mathcal{S}}\,,\quad \hat{\boldsymbol{\Xi}} = \bar{p}\hat{\boldsymbol{\xi}}\,,$$
$$\hat{\boldsymbol{\Psi}} = \hat{\mathcal{P}} - \frac{1}{3}\triangle\hat{\mathcal{S}} - \bar{p}\hat{\psi}\,,\quad \hat{\boldsymbol{\Phi}} = \hat{\mathcal{E}} + \bar{\rho}\hat{\phi}\,,\quad \hat{\mathbf{Y}} = -(\bar{\rho}+\bar{p})(\hat{\mathcal{L}} + \hat{\mathbf{y}})\,, \tag{5.1.50}$$

the four vector equations

$$\hat{\mathbf{V}}_a = -(\bar{\rho}+\bar{p})(\hat{\mathcal{X}}_a + \hat{\mathbf{v}}_a) - \bar{p}\hat{\mathbf{b}}_a\,,\quad \hat{\mathbf{C}}_a = \hat{\mathcal{V}}_a\,,$$
$$\hat{\mathbf{W}}_a = \bar{p}\hat{\mathbf{w}}_a - \frac{1}{2}\upsilon_{abc}\mathrm{d}^b\hat{\mathcal{V}}^c\,,\quad \hat{\mathbf{B}}_a = -(\bar{\rho}+\bar{p})\hat{\mathcal{X}}_b - \bar{p}\hat{\mathbf{v}}_b\,, \tag{5.1.51}$$

and the tensor equation

$$\hat{\mathbf{Q}}_{ab} = 2\hat{\mathcal{T}}_{ab} - \bar{p}\hat{\mathbf{q}}_{ab}\,. \tag{5.1.52}$$

Note the appearance of an additional term involving $\hat{\mathcal{V}}_a$ in the equation for $\hat{\mathbf{W}}_a$. This term arises from the fact that in the decomposition (5.1.29), the term $\mathrm{d}_a\hat{C}_b$ is not symmetrized, while the energy-momentum tensor (5.1.48) contains the strictly symmetric contribution $\mathrm{d}_{(a}\hat{\mathcal{V}}_{b)}$. The antisymmetric part of the former can be rewritten as

$$\mathrm{d}_{[a}\hat{C}_{b]} = \frac{1}{2}\upsilon_{abc}\upsilon^{dec}\mathrm{d}_d\hat{C}_e \tag{5.1.53}$$

and thus absorbed into a redefinition of $\hat{W}_a + \mathrm{d}_a\hat{\Xi}$. Here the contribution to $\hat{\Xi}$ corresponding to the divergence of the last term vanishes due to the Bianchi identity

$$\mathrm{d}_c(\upsilon^{dec}\mathrm{d}_d\hat{C}_e) = \upsilon^{dec}\mathrm{d}_{[c}\mathrm{d}_{d]}\hat{C}_e = \frac{1}{2}\upsilon^{dec}R^f{}_{ecd}\hat{C}_f = 0\,. \tag{5.1.54}$$

Thus, one obtains only the vector contribution we found above.

5.1.8 Energy-momentum conservation and Bianchi identities

The gauge-invariant perturbed field equations shown in the previous section have 16 components, while they depend on only 12 gauge-invariant tetrad components, while the remaining tetrad components have been eliminated by a gauge symmetry under diffeomorphisms. This difference in the number of equations and variables is accounted for by the Bianchi identities on the gravitational side, and equivalently the covariant energy-momentum conservation on the matter side of the field equations, which yield four conditions satisfied by the field equations. For a generic teleparallel gravity theory, these can be written most succinctly as

$$\overset{\circ}{\nabla}{}^{\nu}E_{\mu\nu} - K^{\nu\rho}{}_{\mu}E_{\nu\rho} = 0\,, \tag{5.1.55}$$

and are geometric identities which are satisfied automatically by the field equations [79]. Decomposing the field equations into their symmetric and antisymmetric parts, and using the fact that the contortion is antisymmetric in its first two indices, they reduce to

$$\overset{\circ}{\nabla}{}^{\nu}E_{(\mu\nu)} = 0\,, \quad \overset{\circ}{\nabla}{}^{\nu}E_{[\mu\nu]} - K^{\nu\rho}{}_{\mu}E_{\nu\rho} = 0\,. \tag{5.1.56}$$

On the matter side, they are complemented by the covariant energy-momentum conservation, which simply reads

$$\overset{\circ}{\nabla}{}^{\nu}\Theta_{\mu\nu} = 0\,, \tag{5.1.57}$$

due to the fact that the energy-momentum tensor is symmetric.

In order to derive the aforementioned conditions on the perturbed field equations, we first perform a perturbative expansion of the Bianchi identity (5.1.55). For the background, this yields the equation

$$\partial_t\mathfrak{N} + 3(\mathfrak{N}+\mathfrak{H})\frac{\partial_t\mathcal{A}}{\mathcal{A}} = 0\,. \tag{5.1.58}$$

For the linear perturbation order, one expresses the perturbed field equations through the perturbation tensor $\mathfrak{E}_{\mu\nu}$, and then applies a split into time and space components, followed by an irreducible decomposition using the components (5.1.29). After a tedious, but straightforward calculation one finds that the resulting equations can be fully expressed in terms of the gauge-invariant quantities we introduced in the previous sections. In particular, one finds the time component

$$\begin{aligned}\mathcal{N}\triangle\hat{\mathbf{J}} - \partial_t\mathcal{A}(\triangle\hat{\mathbf{\Sigma}} + 3\hat{\mathbf{\Phi}} + 3\hat{\mathbf{\Psi}}) - \mathcal{A}\partial_t\hat{\mathbf{\Phi}}&\\ - 3\partial_t\mathcal{A}\mathfrak{H}(\hat{\phi}+\hat{\psi}) - 3\mathcal{A}(\mathfrak{N}+\mathfrak{H})\partial_t\hat{\psi} + \mathcal{A}\mathfrak{N}\partial_t\hat{\phi} + \mathcal{N}\mathfrak{H}\triangle\hat{\mathbf{y}} &= 0\,,\end{aligned} \tag{5.1.59}$$

while the spatial part decomposes into the total divergence of the term

$$\begin{aligned}\mathcal{N}\hat{\mathbf{\Psi}} + \mathcal{N}\triangle\hat{\mathbf{\Sigma}} + 2u^2\mathcal{N}\hat{\mathbf{\Sigma}} + 2\mathcal{N}\mathfrak{a}\hat{\mathbf{\Xi}} + \mathcal{N}\mathfrak{v}(\hat{\mathbf{J}} - \hat{\mathbf{Y}}) - \partial_t\mathcal{A}(\hat{\mathbf{J}} + 3\hat{\mathbf{Y}}) - \mathcal{A}\partial_t\hat{\mathbf{Y}}&\\ - \mathcal{N}\mathfrak{N}\hat{\phi} + \mathcal{N}\mathfrak{H}(\hat{\psi} - \hat{\phi}) + \partial_t\mathcal{A}\mathfrak{N}\hat{\mathbf{y}} - \mathcal{N}\mathfrak{N}\mathfrak{v}\hat{\mathbf{y}} - 2\mathcal{N}\mathfrak{H}\mathfrak{a}\hat{\boldsymbol{\xi}} &= 0\,,\end{aligned} \tag{5.1.60}$$

as well as the divergence-free part

$$\begin{aligned}\mathcal{N}\mathfrak{v}(\hat{\mathbf{B}}_a - \hat{\mathbf{V}}_a) - \partial_t\mathcal{A}(\hat{\mathbf{B}}_a + 3\hat{\mathbf{V}}_a) - \mathcal{A}\partial_t\hat{\mathbf{V}}_a + 2u^2\mathcal{N}\hat{\mathbf{C}}_a + 2\mathcal{N}\mathfrak{a}\hat{\mathbf{W}}_a + \mathcal{N}\upsilon_{abc}\mathrm{d}^b(\hat{\mathbf{W}}^c + \mathfrak{a}\hat{\mathbf{C}}^c - \mathfrak{H}\hat{\mathbf{w}}^c)&\\ + (\partial_t\mathcal{A}\mathfrak{N} - \mathcal{N}\mathfrak{v})(\mathfrak{N}\hat{\mathbf{v}}_a + \mathfrak{H}\hat{\mathbf{b}}_a) - 2\partial_t\mathcal{A}\mathfrak{H}\hat{\mathbf{b}}_b + \mathcal{A}\mathfrak{N}\partial_t\hat{\mathbf{b}}_a - 2\mathcal{N}\mathfrak{H}\mathfrak{a}\hat{\mathbf{w}}_a &= 0\,.\end{aligned} \tag{5.1.61}$$

Here we have made use of the background Bianchi identity (5.1.58) in order to simplify the obtained expressions and cancel gauge-dependent terms.

Finally, one proceeds analogously with the energy-momentum conservation (5.1.57). For the background, one finds the relation

$$\partial_t\bar{\rho} + 3(\bar{\rho} + \bar{p})\frac{\partial_t\mathcal{A}}{\mathcal{A}} = 0\,, \tag{5.1.62}$$

which is simply the well-known continuity equation of the cosmologically symmetric fluid energy-momentum tensor, and which has the same form as the cosmological Bianchi identity (5.1.58). To derive the linear perturbation order, it is most straightforward to use the energy-momentum tensor in the form (5.1.40), and then performs an irreducible decomposition. As for the gravitational part of the field equations, one finds that the resulting equations are fully expressed in terms of gauge-invariant quantities. For the time component, one obtains the energy conservation

$$\mathcal{A}\partial_t\hat{\mathcal{E}} + 3\partial_t\mathcal{A}(\hat{\mathcal{E}} + \hat{\mathcal{P}}) + 3\mathcal{A}(\bar{\rho} + \bar{p})\hat{\psi} + \mathcal{N}(\bar{\rho} + \bar{p})\triangle\hat{\mathcal{L}} = 0\,. \tag{5.1.63}$$

The spatial component, which corresponds to the conservation of momentum, splits into a pure divergence part, which reads

$$\mathcal{A}\partial_t[(\bar{\rho}+\bar{p})(\hat{\mathcal{L}}+\hat{\mathbf{y}})]+4\partial_t\mathcal{A}(\bar{\rho}+\bar{p})(\hat{\mathcal{L}}+\hat{\mathbf{y}})+\mathcal{N}\left(\hat{\mathcal{P}} + \frac{2}{3}\triangle\hat{\mathcal{S}} + 2u^2\hat{\mathcal{S}}\right)-\mathcal{N}(\bar{\rho}+\bar{p})\hat{\phi} = 0\,, \tag{5.1.64}$$

and a divergence-free part

$$\mathcal{A}\partial_t[(\bar{\rho}+\bar{p})(\hat{\mathcal{X}}_a+\hat{\mathbf{v}}_a+\hat{\mathbf{b}}_a)]+4\partial_t\mathcal{A}(\bar{\rho}+\bar{p})(\hat{\mathcal{X}}_a+\hat{\mathbf{v}}_a+\hat{\mathbf{b}}_a)+\frac{1}{2}\mathcal{N}\triangle\hat{\mathcal{V}}_a+u^2\mathcal{N}\hat{\mathcal{V}}_a = 0\,. \tag{5.1.65}$$

These are the well-known conservation equations for the perturbed cosmological energy-momentum tensor, expressed in terms of the tetrad perturbation variables we use here.

It is well known that the Bianchi identities and the energy-momentum conservation equations are related to each other by the gravitational field equations. For the cosmological perturbations, this can easily be seen by inserting the gravitational part of the gauge-invariant field equations given in section 5.1.4 into the Bianchi identities given above. One finds that they indeed yield the energy-momentum conservation equations.

5.1.9 Application to TEGR

As an illustrative example, we used the formalism developed in our work [H5] in order to derive the gravitational part of the perturbed cosmological field equations for the teleparallel equivalent of general relativity (TEGR), whose action is a special case of the action (4.2.43) with

$$\mathcal{F}(\mathbb{T}_1, \mathbb{T}_2, \mathbb{T}_3) = -\frac{1}{4}\mathbb{T}_1 - \frac{1}{2}\mathbb{T}_2 + \mathbb{T}_3\,. \tag{5.1.66}$$

In this case the gravitational part of the field equations reduces to Einstein's equations,

$$\kappa^2 E_{\mu\nu} = \mathring{G}_{\mu\nu} = \mathring{R}_{\mu\nu} - \frac{1}{2}\mathring{R}g_{\mu\nu}\,. \tag{5.1.67}$$

The cosmological dynamics for the background and the perturbations can most conveniently expressed in terms of the conformal Hubble parameter (4.2.27) and the conformal time derivative $' = \mathcal{A}\mathcal{N}^{-1}\partial_t$. The background dynamics is governed by the Friedmann equations

$$\kappa^2\mathcal{A}^2\mathfrak{N} = 3(\mathcal{H}^2 + u^2)\,, \quad \kappa^2\mathcal{A}^2\mathfrak{H} = -(2\mathcal{H}' + \mathcal{H}^2 + u^2)\,, \tag{5.1.68}$$

while the dynamics for the perturbations can be expressed in terms of the gauge-invariant, irreducible components of the perturbed field equations, for which one finds the expressions

$$\kappa^2\mathcal{A}^2\hat{\mathbf{\Phi}} = 3(3\mathcal{H}^2 + u^2)\hat{\phi} - 2\mathcal{H}\triangle\hat{\mathbf{y}} + 6\mathcal{H}\hat{\psi}' - 6u^2\hat{\psi} - 2\triangle\hat{\psi}\,, \tag{5.1.69a}$$

$$\kappa^2\mathcal{A}^2\hat{\mathbf{\Psi}} = (\mathcal{H}^2 + 2\mathcal{H}' + 3u^2)\hat{\psi} - 4\mathcal{H}\hat{\psi}' - 2\hat{\psi}'' - 2(\mathcal{H}^2 + 2\mathcal{H}')\hat{\phi} - 2\mathcal{H}\hat{\phi}' + \triangle\left[\hat{\psi} - \hat{\phi} + 2\mathcal{H}\hat{\mathbf{y}} + \hat{\mathbf{y}}'\right]\,, \tag{5.1.69b}$$

$$\kappa^2\mathcal{A}^2\hat{\mathbf{\Xi}} = -(\mathcal{H}^2 + 2\mathcal{H}' + u^2)\hat{\boldsymbol{\xi}}\,, \tag{5.1.69c}$$

$$\kappa^2\mathcal{A}^2\hat{\mathbf{\Sigma}} = \hat{\phi} - \hat{\psi} - 2\mathcal{H}\hat{\mathbf{y}} - \hat{\mathbf{y}}'\,, \tag{5.1.69d}$$

$$\kappa^2\mathcal{A}^2\hat{\mathbf{J}} = -2\hat{\psi}' + (3\mathcal{H}^2 + u^2)\hat{\mathbf{y}} - 2\mathcal{H}\hat{\phi}\,, \tag{5.1.69e}$$

$$\kappa^2\mathcal{A}^2\hat{\mathbf{Y}} = -2\hat{\psi}' - 2u^2\hat{\mathbf{y}} - 2\mathcal{H}\hat{\phi}\,, \tag{5.1.69f}$$

$$\kappa^2\mathcal{A}^2\hat{\mathbf{C}}_a = -\hat{\mathbf{b}}_a' - \hat{\mathbf{v}}_a' - 2\mathcal{H}(\hat{\mathbf{b}}_a + \hat{\mathbf{v}}_a)\,, \tag{5.1.69g}$$

$$\kappa^2\mathcal{A}^2\hat{\mathbf{W}}_a = \upsilon_{abc}\mathrm{d}^b\left[\frac{1}{2}(\hat{\mathbf{b}}^{c\prime} + \hat{\mathbf{v}}^{c\prime}) + \mathcal{H}(\hat{\mathbf{b}}^c + \hat{\mathbf{v}}^c)\right] - (\mathcal{H}^2 + 2\mathcal{H}' + u^2)\hat{\mathbf{w}}_a\,, \tag{5.1.69h}$$

$$\kappa^2\mathcal{A}^2\hat{\mathbf{B}}_a = 2(\mathcal{H}^2 - \mathcal{H}')\hat{\mathbf{b}}_a + 3\mathcal{H}^2\hat{\mathbf{v}}_a + u^2(\hat{\mathbf{b}}_a + 2\hat{\mathbf{v}}_a) - \frac{1}{2}\triangle(\hat{\mathbf{b}}_a + \hat{\mathbf{v}}_a)\,, \tag{5.1.69i}$$

$$\kappa^2\mathcal{A}^2\hat{\mathbf{V}}_a = 3\mathcal{H}^2\hat{\mathbf{b}}_a + u^2(2\hat{\mathbf{b}}_a - \hat{\mathbf{v}}_a) - \frac{1}{2}\triangle(\hat{\mathbf{b}}_a + \hat{\mathbf{v}}_a)\,, \tag{5.1.69j}$$

$$\kappa^2\mathcal{A}^2\hat{\mathbf{Q}}_{ab} = \hat{\mathbf{q}}_{ab}'' + 2\mathcal{H}\hat{\mathbf{q}}_{ab}' - \triangle\hat{\mathbf{q}}_{ab} + (3u^2 + \mathcal{H}^2 + 2\mathcal{H}')\hat{\mathbf{q}}_{ab}\,. \tag{5.1.69k}$$

One easily checks that these components indeed satisfy the Bianchi identities displayed in section 5.1.8. Further, inserting them into the field equations shown in section 5.1.4, one finds that several equations are solved identically, which correspond to the antisymmetric part $E_{[\mu\nu]} = 0$ of the field equations, which become trivial in TEGR. The remaining equations can be summarized as follows. The most simple is the tensor equation, which reads

$$\hat{\mathbf{q}}_{ab}'' + 2\mathcal{H}\hat{\mathbf{q}}_{ab}' - \triangle\hat{\mathbf{q}}_{ab} + 2u^2\hat{\mathbf{q}}_{ab} = 2\kappa^2\mathcal{A}^2\hat{\mathcal{T}}_{ab}\,. \tag{5.1.70}$$

For the vector components, one finds the independent equations

$$\hat{\mathbf{b}}_a' + \hat{\mathbf{v}}_a' + 2\mathcal{H}(\hat{\mathbf{b}}_a + \hat{\mathbf{v}}_a) = -\kappa^2\mathcal{A}^2\hat{\mathcal{V}}_a \tag{5.1.71}$$

and

$$\frac{1}{2}\triangle(\hat{\mathbf{b}}_a + \hat{\mathbf{v}}_a) - (2\mathcal{H}^2 - 2\mathcal{H}' + u^2)(\hat{\mathbf{b}}_a + \hat{\mathbf{v}}_a) = 2(\mathcal{H}^2 - \mathcal{H}' + u^2)\hat{\mathcal{X}}_a\,. \tag{5.1.72}$$

It is convenient to replace the transverse velocity perturbation $\hat{\mathcal{X}}_a$ by the variable

$$\hat{\mathcal{Q}}_a = (\bar{\rho} + \bar{p})(\hat{\mathcal{X}}_a + \hat{\mathbf{b}}_a + \hat{\mathbf{v}}_a) = 2\frac{\mathcal{H}^2 - \mathcal{H}' + u^2}{\kappa^2\mathcal{A}^2}(\hat{\mathcal{X}}_a + \hat{\mathbf{b}}_a + \hat{\mathbf{v}}_a)\,. \tag{5.1.73}$$

In terms of this new variable, the second equation (5.1.72) takes the form

$$\frac{1}{2}\triangle(\hat{\mathbf{b}}_a + \hat{\mathbf{v}}_a) + u^2(\hat{\mathbf{b}}_a + \hat{\mathbf{v}}_a) = \kappa^2\mathcal{A}^2\hat{\mathcal{Q}}_a \tag{5.1.74}$$

of a Poisson equation. The time evolution follows from the remaining equation (5.1.71), from which one now obtains

$$\begin{aligned}\frac{1}{2}\triangle\hat{\mathcal{V}}_a + u^2\hat{\mathcal{V}}_a &= -\frac{1}{\kappa^2\mathcal{A}^2}\left[\frac{1}{2}\triangle(\hat{\mathbf{b}}'_a + \hat{\mathbf{v}}'_a + 2\mathcal{H}\hat{\mathbf{b}}_a + 2\mathcal{H}\hat{\mathbf{v}}_a) + u^2(\hat{\mathbf{b}}'_a + \hat{\mathbf{v}}'_a + 2\mathcal{H}\hat{\mathbf{b}}_a + 2\mathcal{H}\hat{\mathbf{v}}_a)\right]\\ &= -\frac{1}{\mathcal{A}^2}\left[2\mathcal{H}\mathcal{A}^2\hat{\mathcal{Q}}_a + (\mathcal{A}^2\hat{\mathcal{Q}}_a)'\right]\\ &= -\hat{\mathcal{Q}}'_a - 4\mathcal{H}\hat{\mathcal{Q}}_a\,,\end{aligned} \tag{5.1.75}$$

which is simply the conservation equation of transverse momentum. Finally, for the scalar components one has the equations

$$\kappa^2\mathcal{A}^2\hat{\mathcal{E}} = 6\mathcal{H}\hat{\psi}' - 2\triangle\hat{\psi} - 6u^2\hat{\psi} + 6\mathcal{H}^2\hat{\phi} - 2\mathcal{H}\triangle\hat{\mathbf{y}}\,, \tag{5.1.76a}$$

$$(\mathcal{H}^2 - \mathcal{H}' + u^2)\hat{\mathcal{L}} = \hat{\psi}' + \mathcal{H}\hat{\phi} + \mathcal{H}'\hat{\mathbf{y}} - \mathcal{H}^2\hat{\mathbf{y}}\,, \tag{5.1.76b}$$

$$\kappa^2\mathcal{A}^2\hat{\mathcal{S}} = \hat{\phi} - \hat{\psi} - \hat{\mathbf{y}}' - 2\mathcal{H}\hat{\mathbf{y}}\,, \tag{5.1.76c}$$

$$\kappa^2\mathcal{A}^2\left(\hat{\mathcal{P}} - \frac{\triangle\hat{\mathcal{S}}}{3}\right) = \triangle(\hat{\psi} - \hat{\phi} + 2\left[\mathcal{H}\hat{\mathbf{y}} + \hat{\mathbf{y}}') - 2\mathcal{H}(2\hat{\psi}' + \hat{\phi}') - \hat{\psi}'' + u^2\hat{\psi} - (\mathcal{H}^2 + 2\mathcal{H}')\hat{\phi}\right]. \tag{5.1.76d}$$

The last equation can be simplified by adding the second derivative of the preceding equation, to yield

$$\kappa^2\mathcal{A}^2\left(\hat{\mathcal{P}} + \frac{2}{3}\triangle\hat{\mathcal{S}}\right) = -2\mathcal{H}(2\hat{\psi}' + \hat{\phi}') - 2\hat{\psi}'' + 2u^2\hat{\psi} - 2(\mathcal{H}^2 + 2\mathcal{H}')\hat{\phi}\,, \tag{5.1.77}$$

Finally, it is helpful to remark that $\hat{\mathbf{y}}$ can be absorbed into the remaining variables by a suitable redefinition. Hence, one is left with dynamical equations for the variables $\hat{\phi}, \hat{\psi}, \hat{\mathbf{b}}_a + \hat{\mathbf{v}}_a, \hat{\mathbf{q}}_{ab}$, which correspond to the gauge-invariant perturbations. These equations reproduce the well-known cosmological perturbation of Einstein's equations.

5.2 Gauge-invariant parametrized post-Newtonian formalism

Another application of perturbation theory and symmetry transformations of perturbations can be found in the parametrized post-Newtonian (PPN) formalism [159, 160, 161]. It allows to classify any metric theory of gravity by a set of (usually constant) parameters, which are closely related to experimentally observable quantities, thus providing a possibility to test the gravity theories under consideration. This formalism has the advantage that it is straightforward to apply in several steps, each of which comprises of solving linear partial differential equations at a particular perturbation order. However, these equations may become cumbersome if there is a large number of dynamical fields present in the theory, and carry a gauge invariance, which is solved for only in the last stage of the formalism. In order to overcome these difficulties, we devised a gauge-invariant formalism in our work [H6], which both resolves the necessity to choose a gauge and potentially transform to a different gauge after solving the equations, and simplifies the equations by decomposing them into irreducible components. We now present a brief summary of this formalism. In section 5.2.1, we briefly review the perturbative expansion of the gravitational field variables in the PPN formalism. We then show their behavior under higher-order gauge transformations in section 5.2.2. Based on these transformations, we find a set of gauge-invariant gravitational field variables in section 5.2.3, as well as a gauge-invariant decomposition of the energy-momentum tensor of the source matter in section 5.2.4. We relate these expressions to the standard PPN formalism in section 5.2.5, before applying it to a simple example theory in section 5.2.6.

5.2.1 Post-Newtonian geometry perturbations

An important ingredient of the PPN formalism is the perturbative expansion of all dynamical fields of the considered gravity around a given background, which is defined in a fixed frame of reference, in which the metric is asymptotically flat and the source matter is slow-moving. The background is assumed to solve the vacuum field equations, while perturbations are induced by the source matter. Further, different components of the source matter distribution are assigned with different perturbation orders, which correspond to different terms in the perturbative expansion of the dynamical fields. The most fundamental field is assumed to be the spacetime metric, which governs the motion of test masses and light by its geodesics. Its perturbative expansion is written in the form

$$g_{\mu\nu} = \overset{0}{g}_{\mu\nu} + \overset{1}{g}_{\mu\nu} + \overset{2}{g}_{\mu\nu} + \overset{3}{g}_{\mu\nu} + \overset{4}{g}_{\mu\nu} + \mathcal{O}(5)\,, \tag{5.2.1}$$

where the background is assumed to have maximal symmetry under the Poincaré group, and thus must given by the Minkowski metric, $\overset{0}{g}_{\mu\nu} = \eta_{\mu\nu}$. The perturbation orders, which we denoted by overscript numbers, are interpreted as orders of the typical velocity of test masses and matter source constituents in a fixed system of reference. This fixed system of reference is also used in order to perform a $3+1$ split of tensor components. Based on a perturbative expansion of the geodesic equation, which describes the motion of test masses, and the assumption that velocities are small compared to the speed of light in the chosen reference system, only particular components are non-vanishing and relevant in the perturbative expansion. For the metric, these are the components

$$\overset{2}{g}_{00}\,, \quad \overset{2}{g}_{ij}\,, \quad \overset{3}{g}_{i0}\,, \quad \overset{4}{g}_{00}\,, \quad \overset{4}{g}_{ij}\,. \tag{5.2.2}$$

Note that $\overset{4}{g}_{ij}$ is usually not considered in the standard PPN formalism [159]. However, in general it may couple to other fourth-order terms such as $\overset{4}{g}_{00}$, and so it cannot be neglected in general, but needs to be eliminated from the field equations to calculate other relevant components. Further, it may also be considered as a relevant component by itself, which enters as a higher order correction in light deflection [144, 145].

In the case of a teleparallel geometry, considering the tetrad as a fundamental variable instead of the metric, one considers a similar perturbative expansion of the form [158]

$$\theta^A{}_\mu = \overset{0}{\theta}{}^A{}_\mu + \overset{1}{\theta}{}^A{}_\mu + \overset{2}{\theta}{}^A{}_\mu + \overset{3}{\theta}{}^A{}_\mu + \overset{4}{\theta}{}^A{}_\mu + \mathcal{O}(5)\,. \tag{5.2.3}$$

As for the cosmological perturbations discussed in section 5.1, it is most convenient to impose the Weitzenböck gauge condition $\overset{\bullet}{\omega}{}^A{}_{B\mu} \equiv 0$ at all perturbation orders, so that the geometry is fully determined by the tetrad only. For the background one demands again maximal symmetry under the Poincaré group, which allows using the diagonal tetrad

$$\overset{0}{\theta}{}^A{}_\mu = \Delta^A{}_\mu = \mathrm{diag}(1,1,1,1)\,, \tag{5.2.4}$$

which is required to be a solution of the vacuum field equations. For the perturbations, one then introduces a pure spacetime index expression

$$\overset{n}{\theta}_{\mu\nu} = \eta_{AB}\Delta^A{}_\mu \overset{n}{\theta}{}^B{}_\nu\,, \tag{5.2.5}$$

again in analogy to the cosmological perturbation theory. Similarly to the metric, only certain components must be considered, which are given by

$$\overset{2}{\theta}_{00}\,,\quad \overset{2}{\theta}_{ij}\,,\quad \overset{3}{\theta}_{i0}\,,\quad \overset{3}{\theta}_{0j}\,,\quad \overset{4}{\theta}_{00}\,,\quad \overset{4}{\theta}_{ij}\,. \tag{5.2.6}$$

Note that in contrast to the metric, $\theta_{\mu\nu}$ is not necessarily symmetric, so that two index combinations must be considered at the third velocity order.

Finally, it is worth mentioning that also for an independent connection, which appears in the metric-affine geometry, a perturbative expansion of the form

$$\Gamma^{\mu}{}_{\nu\rho} = \overset{0}{\Gamma}{}^{\mu}{}_{\nu\rho} + \overset{1}{\Gamma}{}^{\mu}{}_{\nu\rho} + \overset{2}{\Gamma}{}^{\mu}{}_{\nu\rho} + \overset{3}{\Gamma}{}^{\mu}{}_{\nu\rho} + \overset{4}{\Gamma}{}^{\mu}{}_{\nu\rho} + \mathcal{O}(5) \tag{5.2.7}$$

may be introduced. Here it is important to note that the background $\overset{0}{\Gamma}{}^{\mu}{}_{\nu\rho}$ is given by the coefficients of a connection, while the perturbations are tensor fields. The choice of the background is once more constrained by demanding maximal symmetry under the Poincaré group, which uniquely determines it to be given by the Levi-Civita connection of the Minkowski background. It follows that the background connection is flat, symmetric and compatible with the background metric, and that its connection coefficients $\overset{0}{\Gamma}{}^{\mu}{}_{\nu\rho}$ vanish in Cartesian coordinates. For the perturbation, one finds that the relevant components are given by

$$\overset{2}{\Gamma}{}^{i}{}_{jk}\,,\ \overset{2}{\Gamma}{}^{0}{}_{0k}\,,\ \overset{2}{\Gamma}{}^{0}{}_{j0}\,,\ \overset{2}{\Gamma}{}^{i}{}_{00}\,,\ \overset{3}{\Gamma}{}^{0}{}_{jk}\,,\ \overset{3}{\Gamma}{}^{i}{}_{0k}\,,\ \overset{3}{\Gamma}{}^{i}{}_{j0}\,,\ \overset{3}{\Gamma}{}^{0}{}_{00}\,,\ \overset{4}{\Gamma}{}^{i}{}_{jk}\,,\ \overset{4}{\Gamma}{}^{0}{}_{0k}\,,\ \overset{4}{\Gamma}{}^{0}{}_{j0}\,,\ \overset{4}{\Gamma}{}^{i}{}_{00}\,. \tag{5.2.8}$$

If the connection is derived from another geometric field, such as the Levi-Civita connection of the metric or the Weitzenböck connection in the teleparallel case, several of the aforementioned components may vanish.

5.2.2 Higher order gauge transformations

The fixed frame of reference in which the perturbations of the geometry listed above are defined is not fully determined by the conditions of slow-moving source matter and an asymptotically flat metric. These conditions remain satisfied if one performs a coordinate transformation, which is generated by a vector field X^{μ} whose components exhibit a similar perturbative form as the geometry perturbations, so that the general form of their perturbative expansion is retained. This leads to the condition that the vector field has components

$$\overset{2}{X}{}^{i}\,,\quad \overset{3}{X}{}^{0}\,,\quad \overset{4}{X}{}^{i} \tag{5.2.9}$$

at the respective velocity orders. Further, since one considers higher order perturbations of the fields defining the geometry, also the diffeomorphism generated by the vector field must be expanded beyond the commonly considered linear order. The general form of such a transformation is given by a so-called "knight diffeomorphism" [19, 156, 20]. For the purpose of the PPN formalism, it turns out to be sufficient to expand it up to the quadratic order in the components of the vector field, which yields the explicit formula

$$x'^{\mu} = x^{\mu} + X^{\mu} + \frac{1}{2}X^{\nu}\partial_{\nu}X^{\mu} + \mathcal{O}([X]^3)\,, \tag{5.2.10}$$

generalizing the linear coordinate transformation (5.1.15). It follows that the metric, as any other tensor field, obeys the transformation law

$$g_{\mu\nu} = g'_{\mu\nu} + (\mathcal{L}_X g')_{\mu\nu} + \frac{1}{2}(\mathcal{L}_X\mathcal{L}_X g')_{\mu\nu} + \mathcal{O}([X]^3) \tag{5.2.11}$$

under this change of coordinates, which is reminiscent of a Taylor expansion. Expanding both sides of this equation into the post-Newtonian perturbation orders, it follows that the metric perturbations transform as

$$\overset{2}{g}_{00} = \overset{2}{g}'_{00}\,, \tag{5.2.12a}$$

$$\overset{2}{g}_{ij} = \overset{2}{g}'_{ij} + 2\partial_{(i}\overset{2}{X}_{j)}\,, \tag{5.2.12b}$$

$$\overset{3}{g}_{0i} = \overset{3}{g}'_{0i} + \partial_i\overset{3}{X}_0 + \partial_0\overset{2}{X}_i\,, \tag{5.2.12c}$$

$$\overset{4}{g}_{00} = \overset{4}{g}'_{00} + 2\partial_0\overset{3}{X}_0 + \overset{2}{X}_i\partial_i\overset{2}{g}'_{00}\,, \tag{5.2.12d}$$

$$\overset{4}{g}_{ij} = \overset{4}{g}'_{ij} + 2\partial_{(i}\overset{4}{X}_{j)} + 2\overset{2}{g}'_{k(i}\partial_{j)}\overset{2}{X}_k + \overset{2}{X}_k\partial_k\overset{2}{g}'_{ij} + \partial_{(i}(\overset{2}{X}_{|k}\partial_{k|}\overset{2}{X}_{j)}) + \partial_i\overset{2}{X}_k\partial_j\overset{2}{X}_k\,, \tag{5.2.12e}$$

where the indices of the generating vector field have been lowered using the background Minkowski metric, as it is also commonplace for linear perturbations. Similarly, one derives the transformation of the tetrad perturbations from the analogue formula

$$\theta^A{}_\mu = \theta'^A{}_\mu + (\mathcal{L}_X\theta')^A{}_\mu + \frac{1}{2}(\mathcal{L}_X\mathcal{L}_X\theta')^A{}_\mu + \mathcal{O}([X]^3) \tag{5.2.13}$$

Here it must be taken into account that the tetrad transforms as a one-form. One therefore finds that the perturbation components $\overset{n}{\theta}_{\mu\nu}$, which are defined with respect to a fixed background tetrad, do not transform as perturbations of a tensor field of rank 2, but instead satisfy the transformation rules

$$\overset{2}{\theta}_{00} = \overset{2}{\theta}'_{00}\,, \tag{5.2.14a}$$

$$\overset{2}{\theta}_{ij} = \overset{2}{\theta}'_{ij} + \partial_j\overset{2}{X}_i\,, \tag{5.2.14b}$$

$$\overset{3}{\theta}_{0i} = \overset{3}{\theta}'_{0i} + \partial_i\overset{3}{X}_0\,, \tag{5.2.14c}$$

$$\overset{3}{\theta}_{i0} = \overset{3}{\theta}'_{i0} + \partial_0\overset{2}{X}_i\,, \tag{5.2.14d}$$

$$\overset{4}{\theta}_{00} = \overset{4}{\theta}'_{00} + \partial_0\overset{3}{X}_0 + \overset{2}{X}_i\partial_i\overset{2}{\theta}'_{00}\,, \tag{5.2.14e}$$

$$\overset{4}{\theta}_{ij} = \overset{4}{\theta}'_{ij} + \partial_j\overset{4}{X}_i + \partial_j\overset{2}{X}_k\overset{2}{\theta}'_{ik} + \overset{2}{X}_k\partial_k\overset{2}{\theta}'_{ij} + \frac{1}{2}\partial_j(\overset{2}{X}_k\partial_k\overset{2}{X}_i)\,, \tag{5.2.14f}$$

which are obtained from those of the tetrad perturbation $\overset{n}{\theta}{}^A{}_\mu$. Finally, also for the connection coefficients the transformation law takes the form

$$\Gamma^\mu{}_{\nu\rho} = \Gamma'^\mu{}_{\nu\rho} + (\mathcal{L}_X\Gamma')^\mu{}_{\nu\rho} + \frac{1}{2}(\mathcal{L}_X\mathcal{L}_X\Gamma')^\mu{}_{\nu\rho} + \mathcal{O}([X]^3)\,, \tag{5.2.15}$$

taking into account that $\Gamma^\mu{}_{\nu\rho}$ are not the components of a tensor field, but connection coefficients, and so their Lie derivative is given by the inhomogeneous relation (3.1.10). Decomposing this transformation behavior into velocity orders, one finds the transformation rules

$$\overset{2}{\Gamma}{}^i{}_{00} = \overset{2}{\Gamma}{}'^i{}_{00}\,, \tag{5.2.16a}$$

$$\overset{2}{\Gamma}{}^0{}_{j0} = \overset{2}{\Gamma}{}'^0{}_{j0}\,, \tag{5.2.16b}$$

$$\overset{2}{\Gamma}{}^0{}_{0k} = \overset{2}{\Gamma}{}'^0{}_{0k}\,, \tag{5.2.16c}$$

$$\overset{2}{\Gamma}{}^i{}_{jk} = \overset{2}{\Gamma}{}'^i{}_{jk} + \partial_j\partial_k\overset{2}{X}{}^i\,, \tag{5.2.16d}$$

$$\overset{3}{\Gamma}{}^0{}_{00} = \overset{3}{\Gamma}{}'^0{}_{00}\,, \tag{5.2.16e}$$

$$\overset{3}{\Gamma}{}^i{}_{j0} = \overset{3}{\Gamma}{}'^i{}_{j0} + \partial_0\partial_j\overset{2}{X}{}^i\,, \tag{5.2.16f}$$
$$\overset{3}{\Gamma}{}^i{}_{0k} = \overset{3}{\Gamma}{}'^i{}_{0k} + \partial_0\partial_k\overset{2}{X}{}^i\,, \tag{5.2.16g}$$
$$\overset{3}{\Gamma}{}^0{}_{jk} = \overset{3}{\Gamma}{}'^0{}_{jk} + \partial_j\partial_k\overset{3}{X}{}^0\,, \tag{5.2.16h}$$
$$\overset{4}{\Gamma}{}^i{}_{00} = \overset{4}{\Gamma}{}'^i{}_{00} + \partial_0\partial_0\overset{2}{X}{}^i + \overset{2}{X}{}^l\partial_l\overset{2}{\Gamma}{}^i{}_{00} - \overset{2}{\Gamma}{}^l{}_{00}\partial_l\overset{2}{X}{}^i\,, \tag{5.2.16i}$$
$$\overset{4}{\Gamma}{}^0{}_{j0} = \overset{4}{\Gamma}{}'^0{}_{j0} + \partial_0\partial_j\overset{3}{X}{}^0 + \overset{2}{X}{}^l\partial_l\overset{2}{\Gamma}{}^0{}_{j0} + \overset{2}{\Gamma}{}^0{}_{l0}\partial_j\overset{2}{X}{}^l\,, \tag{5.2.16j}$$
$$\overset{4}{\Gamma}{}^0{}_{0k} = \overset{4}{\Gamma}{}'^0{}_{0k} + \partial_0\partial_k\overset{3}{X}{}^0 + \overset{2}{X}{}^l\partial_l\overset{2}{\Gamma}{}^0{}_{0k} + \overset{2}{\Gamma}{}^0{}_{0l}\partial_k\overset{2}{X}{}^l\,, \tag{5.2.16k}$$
$$\begin{aligned}\overset{4}{\Gamma}{}^i{}_{jk} &= \overset{4}{\Gamma}{}'^i{}_{jk} + \partial_j\partial_k\overset{4}{X}{}^i + \overset{2}{X}{}^l\partial_l\overset{2}{\Gamma}{}^i{}_{jk} - \partial_l\overset{2}{X}{}^i\overset{2}{\Gamma}{}^l{}_{jk} + \partial_j\overset{2}{X}{}^l\overset{2}{\Gamma}{}^i{}_{lk} + \partial_k\overset{2}{X}{}^l\overset{2}{\Gamma}{}^i{}_{jl}\\ &\quad + \frac{1}{2}\left(\overset{2}{X}{}^l\partial_j\partial_k\partial_l\overset{2}{X}{}^i - \partial_l\overset{2}{X}{}^i\partial_j\partial_k\overset{2}{X}{}^l + \partial_j\overset{2}{X}{}^l\partial_k\partial_l\overset{2}{X}{}^i + \partial_k\overset{2}{X}{}^l\partial_j\partial_l\overset{2}{X}{}^i\right).\end{aligned} \tag{5.2.16l}$$

Assuming that the field equations of the gravity theory under consideration are invariant under diffeomorphisms, it follows that the gauge transformations shown above transform perturbative solutions to the field equations again into perturbative solutions. Hence, the solution is determined only up to the components (5.2.9) of the gauge vector fields, which correspond to the choice of the coordinate system. Any physically meaningful quantities, however, must be independent of the choice of coordinates, and so one needs to split the fundamental field variables into gauge-invariant physical quantities, which retain their form independently of the choice of the coordinate system, and pure gauge variables, which encode this choice, but carry no physical meaning. This is shown in the following sections.

5.2.3 Gauge-invariant geometry perturbations

The common procedure to obtain gauge-invariant perturbations of the fundamental, geometric field variables is to specify a particular, fixed gauge, in which certain conditions on the perturbations are imposed, which fix as many components of the geometric field perturbations as there are gauge degrees of freedom in the gauge vector fields (5.2.9). The remaining perturbation components, which are not restricted by the gauge choice, are promoted to gauge-invariant variables, i.e., one expresses the perturbations in any other gauge through the components in the fixed gauge and the components of the gauge transformation relating the different gauges, as shown in the context of cosmological perturbations in section 5.1.3. In the PPN formalism, the most common choice for this distinguished gauge is the so-called standard PPN gauge [159], whose metric components we denote by ${}^{\mathcal{P}}g_{\mu\nu}$, and which is chosen such that the metric attains a simple relation with a set of Poisson-like integrals of the matter source. However, from a geometric point of view, there are more convenient gauge choices, which lead to a simpler form of the perturbative field equations, and thus simplify the procedure of solving them. In our work [H6], we introduced two such gauges, in which either the components of the metric or the tetrad simplify, so that one may chose the appropriate gauge according to the fundamental field variable of the gravity theory under consideration. This is done by introducing a similar irreducible decomposition as in the case of linear perturbations in cosmology. Applying this decomposition to the gauge vector field (5.2.9), we decompose the non-vanishing components according to

$$\overset{k}{X}_i = \partial_i\overset{k}{X}{}^{\blacklozenge} + \overset{k}{X}{}^{\diamond}_i\,, \quad \overset{k}{X}_0 = \overset{k}{X}{}^{\star}\,, \tag{5.2.17}$$

at any perturbation order k, where $\partial^i\overset{k}{X}{}^{\diamond}_i = 0$, and indices have been raised and lowered with the background Minkowski metric. Hence, the relevant components that constitute a gauge transformation are given by

$$\overset{2}{X}^{\blacklozenge}\,, \quad \overset{2}{X}{}^{\diamond}_{i}\,, \quad \overset{3}{X}{}^{\star}\,, \quad \overset{4}{X}{}^{\blacklozenge}\,, \quad \overset{4}{X}{}^{\diamond}_{i}\,. \tag{5.2.18}$$

Taking into account that a divergence-free vector has only two independent components, these contain three independent components at each even velocity order and one independent component at the odd velocity order, as it is also the case for the components (5.2.9) before the decomposition.

A similar decomposition is then applied to all geometric field variables, taking into account that certain components can be eliminated by a suitable gauge transformation. For the metric, which is subject to the transformation rules (5.2.12), this allows to use a gauge $\mathcal{M}$ as follows. First, we note that by a suitable choice of $\overset{k}{X}_i$ it is possible to eliminate certain components of ${}^{\mathcal{M}}\overset{k}{g}_{ij}$, such that only a diagonal (pure trace) and a trace-free, divergence-free part remain. Similarly, we may choose $\overset{k}{X}_0$ such that any divergence is eliminated from ${}^{\mathcal{M}}\overset{k}{g}_{0i}$, and retain only a divergence-free part. These components thus take the form

$${}^{\mathcal{M}}\overset{k}{g}_{00} = \overset{k}{\mathbf{g}}{}^{\star}\,, \quad {}^{\mathcal{M}}\overset{k}{g}_{0i} = \overset{k}{\mathbf{g}}{}^{\diamond}_{i}\,, \quad {}^{\mathcal{M}}\overset{k}{g}_{ij} = \overset{k}{\mathbf{g}}{}^{\bullet}\delta_{ij} + \overset{k}{\mathbf{g}}{}^{\dagger}_{ij}\,, \tag{5.2.19}$$

where the irreducible components satisfy the constraint equations

$$\partial^i \overset{k}{\mathbf{g}}{}^{\diamond}_{i} = 0\,, \quad \partial^i \overset{k}{\mathbf{g}}{}^{\dagger}_{ij} = 0\,, \quad \overset{k}{\mathbf{g}}{}^{\dagger}_{[ij]} = 0\,, \quad \overset{k}{\mathbf{g}}{}^{\dagger}_{ii} = 0\,. \tag{5.2.20}$$

Keeping in mind that only the components (5.2.2) are non-vanishing in the PPN formalism, one thus finds that the non-vanishing gauge-invariant variables are given by

$$\overset{2}{\mathbf{g}}{}^{\star}\,, \quad \overset{2}{\mathbf{g}}{}^{\bullet}\,, \quad \overset{2}{\mathbf{g}}{}^{\dagger}_{ij}\,, \quad \overset{3}{\mathbf{g}}{}^{\diamond}_{i}\,, \quad \overset{4}{\mathbf{g}}{}^{\star}\,, \quad \overset{4}{\mathbf{g}}{}^{\bullet}\,, \quad \overset{4}{\mathbf{g}}{}^{\dagger}_{ij}\,. \tag{5.2.21}$$

Counting the number of independent components at each velocity order, where the tensor component $\overset{k}{\mathbf{g}}{}^{\dagger}_{ij}$ contains two independent components, we find that they have less independent components than the perturbations (5.2.2) before the decomposition, and that their difference is accounted for by the pure gauge variables (5.2.18) at each velocity order. Hence, we have fully decomposed the general metric perturbations into gauge-invariant and pure gauge variables.

While the aforementioned decomposition of the metric is most convenient to calculate the post-Newtonian limit in gravity theories whose fundamental field variable is the metric, for tetrad-based gravity theories it is more convenient to consider and decompose the tetrad perturbations (5.2.6) instead. As we will see later, it turns out to be more practical to first decompose the tetrad perturbation into its symmetric and antisymmetric parts, which we denote by

$$\overset{k}{\theta}_{\mu\nu} = \overset{k}{s}_{\mu\nu} + \overset{k}{a}_{\mu\nu}\,, \quad \overset{k}{s}_{\mu\nu} = \overset{k}{\theta}_{(\mu\nu)}\,, \quad \overset{k}{a}_{\mu\nu} = \overset{k}{\theta}_{[\mu\nu]}\,. \tag{5.2.22}$$

Then we introduce a gauge $\mathcal{T}$, in which certain components of the tetrad perturbations are eliminated through a suitable choice of the gauge vector field (5.2.9). In this gauge the tetrad perturbation reads

$${}^{\mathcal{T}}\overset{k}{s}_{00} = \overset{k}{\boldsymbol{\theta}}{}^{\star}\,, \quad {}^{\mathcal{T}}\overset{k}{s}_{0i} = \overset{k}{\boldsymbol{\theta}}{}^{\diamond}_{i}\,, \quad {}^{\mathcal{T}}\overset{k}{s}_{ij} = \overset{k}{\boldsymbol{\theta}}{}^{\bullet}\delta_{ij} + \overset{k}{\boldsymbol{\theta}}{}^{\dagger}_{ij}\,, \quad {}^{\mathcal{T}}\overset{k}{a}_{0i} = \partial_i\overset{k}{\boldsymbol{\theta}}{}^{\blacklozenge} + \overset{k}{\boldsymbol{\theta}}{}^{\circ}_{i}\,, \quad {}^{\mathcal{T}}\overset{k}{a}_{ij} = \epsilon_{ijk}(\partial_k\overset{k}{\boldsymbol{\theta}}{}^{\blacksquare} + \overset{k}{\boldsymbol{\theta}}{}^{\square}_{k})\,, \tag{5.2.23}$$

where the gauge-invariant irreducible components satisfy

$$\partial^i \overset{k}{\boldsymbol{\theta}}{}^{\diamond}_{i} = \partial^i \overset{k}{\boldsymbol{\theta}}{}^{\circ}_{i} = \partial^i \overset{k}{\boldsymbol{\theta}}{}^{\square}_{i} = 0\,, \quad \partial^i \overset{k}{\boldsymbol{\theta}}{}^{\dagger}_{ij} = 0\,, \quad \overset{k}{\boldsymbol{\theta}}{}^{\dagger}_{[ij]} = 0\,, \quad \overset{k}{\boldsymbol{\theta}}{}^{\dagger}_{ii} = 0\,. \tag{5.2.24}$$

Also for the tetrad only certain components of the perturbations are non-vanishing. By comparison with the components (5.2.6), we see that the only gauge-invariant components to be considered are given by

$$\overset{2}{\boldsymbol{\theta}}{}^{\star}\,, \quad \overset{2}{\boldsymbol{\theta}}{}^{\bullet}\,, \quad \overset{2}{\boldsymbol{\theta}}{}^{\blacksquare}\,, \quad \overset{2}{\boldsymbol{\theta}}{}^{\square}_{i}\,, \quad \overset{2}{\boldsymbol{\theta}}{}^{\dagger}_{ij}\,, \quad \overset{3}{\boldsymbol{\theta}}{}^{\blacklozenge}\,, \quad \overset{3}{\boldsymbol{\theta}}{}^{\diamond}_{i}\,, \quad \overset{3}{\boldsymbol{\theta}}{}^{\circ}_{i}\,, \quad \overset{4}{\boldsymbol{\theta}}{}^{\star}\,, \quad \overset{4}{\boldsymbol{\theta}}{}^{\bullet}\,, \quad \overset{4}{\boldsymbol{\theta}}{}^{\blacksquare}\,, \quad \overset{4}{\boldsymbol{\theta}}{}^{\square}_{i}\,, \quad \overset{4}{\boldsymbol{\theta}}{}^{\dagger}_{ij}\,. \tag{5.2.25}$$

Again one finds that the number of independent components of these gauge-invariant variables is complemented by the number of components of the gauge vector field (5.2.9), so that their sum matches the number of components of the original perturbations (5.2.6) at each velocity order.

It follows from their similar definition that the metric gauge $\mathcal{M}$ and tetrad gauge $\mathcal{T}$ are closely related to each other, yet they are not identical. This can be seen by expanding the relation (2.2.22) between the metric and the tetrad into velocity orders, to obtain

$$\overset{2}{g}_{00} = 2\overset{2}{\theta}_{00}\,, \quad \overset{2}{g}_{ij} = 2\overset{2}{\theta}_{(ij)}\,, \quad \overset{3}{g}_{0i} = 2\overset{3}{\theta}_{(0i)}\,, \quad \overset{4}{g}_{00} = -(\overset{2}{\theta}_{00})^2 + 2\overset{4}{\theta}_{00}\,, \quad \overset{4}{g}_{ij} = 2\overset{4}{\theta}_{(ij)} + \overset{2}{\theta}_{ki}\overset{2}{\theta}_{kj}\,. \tag{5.2.26}$$

One finds that the gauge conditions at the second and third velocity order, which determine the components $\overset{2}{X}{}^{i}$ and $\overset{3}{X}{}^{0}$ of the gauge vector fields, are equivalent for the metric components ${}^{\mathcal{M}}\overset{2}{g}_{ij}$ and ${}^{\mathcal{M}}\overset{3}{g}_{0i}$ and the symmetric tetrad components ${}^{\mathcal{T}}\overset{2}{s}_{ij}$ and ${}^{\mathcal{T}}\overset{3}{s}_{0i}$. Hence, the two gauges, and therefore also the components of the geometric field perturbations, agree up to the third velocity order. Further, as one can see from the transformations (5.2.12) and (5.2.14), the fourth order time components $\overset{4}{g}_{00}$ and $\overset{4}{\theta}_{00}$ depend only on the second and third order gauge choice, and therefore also agree. However, at the fourth velocity order, the gauge conditions for the two gauges disagree due to the appearance of the non-linear term $\overset{2}{\theta}_{ki}\overset{2}{\theta}_{kj}$ in the relation (5.2.26), so that also the fourth order gauge vector field $\overset{4}{X}{}^{i}$ disagrees, which enters the fourth order spatial components of the metric and the tetrad. This means that the choice of either gauge must explicitly taken into account in the calculation of higher-order effects [144, 145]. However, for the remaining metric and tetrad components, which are used in the standard PPN formalism, the two gauges agree, and so one may choose either of the two gauges in order to obtain the same result for these components. In the remainder of this section, only these components will be considered, so that the formulas given hold in both gauges. Further, we will denote all components of tensorial quantities in this chosen gauge by boldface symbols.

5.2.4 Gauge-invariant matter source

In order to solve the perturbative field equations of a given gravity theory in the gauge-invariant formalism, the irreducible decomposition of the geometric quantities describing the gravitational field must be complemented by a corresponding decomposition of the energy-momentum tensor. In the PPN formalism, this matter source is assumed to be a perfect fluid, which is described in a $3+1$ split of spacetime by its density $\boldsymbol{\rho}$, pressure $\mathbf{p}$, specific internal energy $\boldsymbol{\Pi}$ and velocity $\mathbf{v}^i$, where we used boldface symbols to indicate that we express these quantities in the chosen gauge discussed in the previous section. The energy-momentum tensor then takes the general form

$$\boldsymbol{\Theta}^{\star} = \boldsymbol{\Theta}_{00} = \boldsymbol{\rho}\left(1 - \overset{2}{\mathbf{g}}_{00} + \mathbf{v}^2 + \boldsymbol{\Pi}\right) + \mathcal{O}(6)\,, \tag{5.2.27a}$$

$$\boldsymbol{\Theta}^{\diamond}_{i} + \partial_i\boldsymbol{\Theta}^{\blacklozenge} = \boldsymbol{\Theta}_{0i} = -\boldsymbol{\rho}\mathbf{v}_i + \mathcal{O}(5)\,, \tag{5.2.27b}$$

$$\boldsymbol{\Theta}^{\bullet}\delta_{ij} + \triangle_{ij}\boldsymbol{\Theta}^{\blacktriangle} + 2\partial_{(i}\boldsymbol{\Theta}^{\triangle}_{j)} + \boldsymbol{\Theta}^{\dagger}_{ij} = \boldsymbol{\Theta}_{ij} = \boldsymbol{\rho}\mathbf{v}_i\mathbf{v}_j + \mathbf{p}\delta_{ij} + \mathcal{O}(6)\,, \tag{5.2.27c}$$

where we have performed a decomposition into irreducible components on the left-hand side. As for the metric components, they are subject to the conditions

$$\partial^i \mathbf{\Theta}^{\diamond}_i = \partial^i \mathbf{\Theta}^{\triangle}_i = 0\,, \quad \partial^i \mathbf{\Theta}^{\dagger}_{ij} = 0\,, \quad \mathbf{\Theta}^{\dagger}_{[ij]} = 0\,, \quad \mathbf{\Theta}^{\dagger}_{ii} = 0\,. \tag{5.2.28}$$

Together with the irreducible decomposition of the geometry perturbations, one can thus finally decompose also the gravitational field equations. It is the virtue of the gauge-invariant, irreducible decomposition that the scalar, vector and tensor contributions decouple from each other, which greatly simplified the task of solving the resulting equations.

5.2.5 Parametrized post-Newtonian formalism

The main ingredient of the PPN formalism, besides the perturbative expansion of both gravitational field variables and matter source terms, is a generic, parametrized form of the metric tensor, which accommodates for a wide range of gravity theories on one side, and which contains a number of free, constant parameters, whose values are closely linked to observable quantities in solar system experiments on the other side. Comparing the measured values of these parameters with their values derived from a gravity theory under consideration thus allows testing the predictions of the theory, while avoiding the necessity to calculate the outcome of every experiment in each considered gravity theory. In their most common form, these parameters are denoted

$$\gamma\,, \quad \beta\,, \quad \alpha_1\,, \quad \alpha_2\,, \quad \alpha_3\,, \quad \zeta_1\,, \quad \zeta_2\,, \quad \zeta_3\,, \quad \zeta_4\,, \quad \xi \tag{5.2.29}$$

and measure the spatial curvature generated by gravity, the non-linear in the superposition law, as well as violations of local Lorentz invariance, local position invariance and total energy-momentum conservation. The PPN parameters, which are specific to the theory under consideration, but independent of the particular choice of the matter configuration, are complemented by the PPN potentials, which describe the matter source independently of the chosen theory. They are defined as the solutions to the Poisson equations

$$\begin{gathered}
\triangle \mathbf{U} = -4\pi\boldsymbol{\rho}\,, \quad \triangle\triangle\boldsymbol{\mathcal{A}} = 8\pi(\boldsymbol{\rho}\mathbf{v}_i\mathbf{v}_j)_{,ij} - 4\pi\triangle(\boldsymbol{\rho}|\mathbf{v}|^2)\,, \quad \triangle\triangle\boldsymbol{\mathcal{B}} = 8\pi[\triangle\mathbf{p} - (\mathbf{U}_{,i}\boldsymbol{\rho})_{,i}]\,, \\
\triangle\mathbf{\Phi}_1 = -4\pi\boldsymbol{\rho}|\mathbf{v}|^2\,, \quad \triangle\mathbf{\Phi}_2 = -4\pi\boldsymbol{\rho}\mathbf{U}\,, \quad \triangle\mathbf{\Phi}_3 = -4\pi\boldsymbol{\rho}\mathbf{\Pi}\,, \quad \triangle\mathbf{\Phi}_4 = -4\pi\mathbf{p}\,, \\
\triangle\mathbf{V}_i = -4\pi\boldsymbol{\rho}\mathbf{v}_i\,, \quad \triangle\mathbf{W}_i = -4\pi\boldsymbol{\rho}\mathbf{v}_i + 2\mathbf{U}_{,0i}\,, \quad \triangle\mathbf{\Phi}_W = 4\pi\boldsymbol{\rho}\mathbf{U} - 4\mathbf{U}_{,i}\mathbf{U}_{,i} + 2\mathbf{U}_{,ij}\boldsymbol{\chi}_{,ij}\,.
\end{gathered} \tag{5.2.30}$$

While the choice of these potentials may appear arbitrary at first glance, it turns out that they appear in the solutions of a wide range of gravity theories. Moreover, they turn out to be particularly useful in the gauge-invariant PPN formalism, as they allow to explicitly express the relevant and non-vanishing irreducible components of the energy-momentum tensor as

$$\begin{gathered}
\overset{2}{\mathbf{T}}{}^{\star} = \boldsymbol{\rho} = -\frac{1}{4\pi}\triangle\mathbf{U}\,, \quad \overset{3}{\mathbf{T}}{}^{\blacklozenge} = -\frac{1}{4\pi}\partial_0\mathbf{U}\,, \quad \overset{3}{\mathbf{T}}{}^{\diamond}_i = \frac{1}{8\pi}\triangle(\mathbf{V}_i + \mathbf{W}_i)\,, \\
\overset{4}{\mathbf{T}}{}^{\star} = -\frac{1}{4\pi}\triangle\left(\mathbf{\Phi}_3 + \mathbf{\Phi}_1 - 2\mathbf{\Phi}_2\right)\,, \quad \overset{4}{\mathbf{T}}{}^{\bullet} = -\frac{1}{12\pi}\triangle(\mathbf{\Phi}_1 + 3\mathbf{\Phi}_4)\,, \quad \overset{4}{\mathbf{T}}{}^{\blacktriangle} = \frac{1}{16\pi}(3\mathfrak{A} - \mathbf{\Phi}_1)\,.
\end{gathered} \tag{5.2.31}$$

Finally, the PPN parameters and PPN potentials given above are used in order to construct a generic form of the metric tensor, which is usually given in the so-called standard PPN gauge [159]. The irreducible components of this generic metric tensor are given by

$$\overset{2}{\mathbf{g}}{}^{\star} = 2\mathbf{U}\,, \tag{5.2.32a}$$

$$\overset{2}{\mathbf{g}}{}^{\bullet} = 2\gamma\mathbf{U}\,, \tag{5.2.32b}$$

$$\overset{2}{\mathbf{g}}^{\dagger}_{ij} = 0\,, \tag{5.2.32c}$$

$$\overset{3}{\mathbf{g}}^{\diamond}_{i} = -\left(1 + \gamma + \frac{\alpha_1}{4}\right)(\mathbf{V}_i + \mathbf{W}_i)\,, \tag{5.2.32d}$$

$$\begin{aligned}\overset{4}{\mathbf{g}}^{\star} &= \frac{1}{2}(2 - \alpha_1 + 2\alpha_2 + 2\alpha_3)\mathbf{\Phi}_1 + 2(1 + 3\gamma - 2\beta + \zeta_2 + \xi)\mathbf{\Phi}_2 \\ &\quad + 2(1 + \zeta_3)\mathbf{\Phi}_3 + 2(3\gamma + 3\zeta_4 - 2\xi)\mathbf{\Phi}_4 - 2\xi\mathbf{\Phi}_W - 2\beta\mathbf{U}^2 \\ &\quad + \frac{1}{2}(2 + 4\gamma + \alpha_1 - 2\alpha_2)\mathfrak{A} + \frac{1}{2}(2 + 4\gamma + \alpha_1 - 2\alpha_2 + 2\zeta_1 - 4\xi)\mathfrak{B}\,. \end{aligned} \tag{5.2.32e}$$

Alternatively, one may also use the relation (5.2.26) between the perturbative expansions of the metric and the tetrad, and use a generic PPN form of the latter instead. Note the appearance of the symmetric tetrad perturbation $\overset{k}{\theta}_{(\mu\nu)}$ in these relations, which is the reason for choosing the particular form (5.2.23) of the irreducible decomposition of the tetrad, as it yields a one-to-one correspondence between the metric components (5.2.32) given above and the tetrad components

$$\overset{2}{\boldsymbol{\theta}}^{\star} = \mathbf{U}\,, \tag{5.2.33a}$$

$$\overset{2}{\boldsymbol{\theta}}^{\bullet} = \gamma\mathbf{U}\,, \tag{5.2.33b}$$

$$\overset{2}{\boldsymbol{\theta}}^{\dagger}_{ij} = 0\,, \tag{5.2.33c}$$

$$\overset{3}{\boldsymbol{\theta}}^{\diamond}_{i} = -\frac{1}{2}\left(1 + \gamma + \frac{\alpha_1}{4}\right)(\mathbf{V}_i + \mathbf{W}_i)\,, \tag{5.2.33d}$$

$$\begin{aligned}\overset{4}{\boldsymbol{\theta}}^{\star} &= \frac{1}{4}(2 - \alpha_1 + 2\alpha_2 + 2\alpha_3)\mathbf{\Phi}_1 + (1 + 3\gamma - 2\beta + \zeta_2 + \xi)\mathbf{\Phi}_2 \\ &\quad + (1 + \zeta_3)\mathbf{\Phi}_3 + (3\gamma + 3\zeta_4 - 2\xi)\mathbf{\Phi}_4 - \xi\mathbf{\Phi}_W + \frac{1}{2}(1 - 2\beta)\mathbf{U}^2 \\ &\quad + \frac{1}{4}(2 + 4\gamma + \alpha_1 - 2\alpha_2)\mathfrak{A} + \frac{1}{4}(2 + 4\gamma + \alpha_1 - 2\alpha_2 + 2\zeta_1 - 4\xi)\mathfrak{B}\,. \end{aligned} \tag{5.2.33e}$$

Hence, using either set of fundamental variables, the task of determining the post-Newtonian limit reduces to calculating a small number of irreducible components by solving the gravitational field equations.

5.2.6 Application to scalar-tensor gravity

In our work [H6], we finally demonstrate the usage of the gauge-invariant PPN formalism by applying it to a simple scalar-tensor gravity theory, whose action is given by [136]

$$S = \frac{1}{2\kappa^2}\int_M \mathrm{d}^4x\sqrt{-g}\left(\psi R - \frac{\omega(\psi)}{\psi}\partial_\rho\psi\partial^\rho\psi\right) + S_m[g_{\mu\nu}, \chi]\,. \tag{5.2.34}$$

The gravitational field variables of this theory are the metric $g_{\mu\nu}$ and scalar field ψ. In addition, the gravitational part of the action is complemented by a matter action S_m, which also depends on the matter fields χ. The dynamics of the scalar field is determined by a free function ω. The vacuum background solution takes the form

$$\overset{0}{g}_{\mu\nu} = \eta_{\mu\nu}\,, \quad \overset{0}{\psi} = \Psi\,, \tag{5.2.35}$$

where the constant background value Ψ of the scalar field is related to the Newtonian gravitational constant κ^2 via the normalization condition

$$\kappa^2 = 4\pi\Psi\frac{2\omega_0 + 3}{\omega_0 + 2}\,. \tag{5.2.36}$$

Here we used the convention

$$\omega_0 = \omega(\Psi)\,, \quad \omega_1 = \omega'(\Psi) \tag{5.2.37}$$

for the Taylor coefficients of ω at the constant background value Ψ. Solving the perturbative field equations one finds that the irreducible components of the metric perturbations take the form

$$\overset{2}{\mathbf{g}}{}^{\star} = 2\mathbf{U}\,, \tag{5.2.38a}$$

$$\overset{2}{\mathbf{g}}{}^{\bullet} = 2\frac{\omega_0+1}{\omega_0+2}\mathbf{U}\,, \tag{5.2.38b}$$

$$\overset{2}{\mathbf{g}}{}^{\dagger}_{ij} = 0\,, \tag{5.2.38c}$$

$$\overset{3}{\mathbf{g}}{}^{\diamond}_{i} = -\frac{2\omega_0+3}{\omega_0+2}(\mathbf{V}_i+\mathbf{W}_i)\,, \tag{5.2.38d}$$

$$\begin{aligned}\overset{4}{\mathbf{g}}{}^{\star} = {}& \frac{3\omega_0+4}{\omega_0+2}(\mathfrak{A}+\mathfrak{B}) - 2\left(1+\frac{\omega_1\Psi}{4(2\omega_0+3)(\omega_0+2)^2}\right)\mathbf{U}^2\\ &+\mathbf{\Phi}_1+\left(\frac{4\omega_0+2}{\omega_0+2}-\frac{\omega_1\Psi}{(2\omega_0+3)(\omega_0+2)^2}\right)\mathbf{\Phi}_2+3\mathbf{\Phi}_3+6\frac{\omega_0+1}{\omega_0+2}\mathbf{\Phi}_4\,.\end{aligned} \tag{5.2.38e}$$

Equivalently, one may express the gravitational field equations through the tetrad instead of the metric. In this case the solution yields the irreducible tetrad perturbation components

$$\overset{2}{\boldsymbol{\theta}}{}^{\star} = \mathbf{U}\,, \tag{5.2.39a}$$

$$\overset{2}{\boldsymbol{\theta}}{}^{\bullet} = \frac{\omega_0+1}{\omega_0+2}\mathbf{U}\,, \tag{5.2.39b}$$

$$\overset{2}{\boldsymbol{\theta}}{}^{\dagger}_{ij} = 0\,, \tag{5.2.39c}$$

$$\overset{3}{\boldsymbol{\theta}}{}^{\diamond}_{i} = -\frac{2\omega_0+3}{2\omega_0+4}(\mathbf{V}_i+\mathbf{W}_i)\,, \tag{5.2.39d}$$

$$\begin{aligned}\overset{4}{\boldsymbol{\theta}}{}^{\star} = {}& \frac{3\omega_0+4}{2\omega_0+4}(\mathfrak{A}+\mathfrak{B}) - \left(\frac{1}{2}+\frac{\omega_1\Psi}{4(2\omega_0+3)(\omega_0+2)^2}\right)\mathbf{U}^2\\ &+\frac{1}{2}\mathbf{\Phi}_1+\left(\frac{2\omega_0+1}{\omega_0+2}-\frac{\omega_1\Psi}{2(2\omega_0+3)(\omega_0+2)^2}\right)\mathbf{\Phi}_2+\frac{3}{2}\mathbf{\Phi}_3+3\frac{\omega_0+1}{\omega_0+2}\mathbf{\Phi}_4\,.\end{aligned} \tag{5.2.39e}$$

Finally, one compares the result with the generic metric perturbation (5.2.32) or its tetrad equivalent (5.2.33). One finds that it indeed has the expected form, where the constant PPN parameters are given by

$$\gamma = \frac{\omega_0+1}{\omega_0+2}\,, \quad \beta = 1+\frac{\omega_1\Psi}{4(2\omega_0+3)(\omega_0+2)^2}\,, \quad \alpha_1=\alpha_2=\alpha_3=\zeta_1=\zeta_2=\zeta_3=\zeta_4=\xi=0\,. \tag{5.2.40}$$

This result is the well-known post-Newtonian limit of the studied scalar-tensor theory [136].

5.3 Non-trivial rotations in Finsler geometry

In the previous sections we have discussed gravity theories in which the dynamical fields are defined as sections of fiber bundles, mostly tensor bundles, whose base manifold is identified as spacetime, so that spacetime symmetries act on this base space. This is different in gravity theories based on Finsler geometry, where the base space is (constructed from) the tangent bundle, and the dynamical fields are most often represented by d-tensors. In both geometric settings, it is common to study background solutions which exhibit spherical symmetry. To discuss perturbations of such spherically symmetric background, it is helpful to decompose such perturbations into irreducible representations of the rotation group. For tensor fields, this leads to so-called tensor spherical harmonics [107, 149]. In our work [H7], which we summarize here, we generalized this concept to d-tensors. In section 5.3.1, we discuss the action of rotations on the tangent bundle via their canonical lift, and introduce a suitable set of coordinates. Using these coordinates, we find it straightforward to construct harmonic functions in section 5.3.2. These are then used to recursively construct harmonic d-tensors in section 5.3.3. While our work [H7] focuses on this extrinsic approach, we remark in section 5.3.4 that also a fully intrinsic approach may be considered.

5.3.1 Rotations in the tangent bundle

As discussed in section 3.1.3, the action of an infinitesimal diffeomorphism generated by a vector field X^μ on the base manifold M of a Finsler geometry is governed by its canonical lift $\hat{X}^\mu$ to the tangent bundle TM. Hence, to study the action of the rotation generators (4.1.1), we need to derive their canonical lifts. While in principle it is straightforward to calculate these canonical lifts in the coordinates on TM which are induced by the spherical coordinates $(t, r, \vartheta, \varphi)$ following the definition (2.2.35), one obtains a rather lengthy result, which leads to complicated symmetry conditions and transformation rules. In order to simplify the result, it is useful to introduce different sets of non-induced coordinates on TM. For simplicity, we omit the time coordinate t in the following considerations, and discuss Finsler geometry on a three-dimensional space only. Starting from the Cartesian coordinates (x^1, x^2, x^3), together with the corresponding induced coordinates $(\bar{x}^1, \bar{x}^2, \bar{x}^3)$ on each tangent space, one can define two more sets of coordinates on TM. The first set $(r, \vartheta, \varphi, \varrho, \alpha, \beta)$, which we call co-rotated spherical coordinates, is defined by

$$\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos\vartheta & 0 & \sin\vartheta \\ 0 & 1 & 0 \\ -\sin\vartheta & 0 & \cos\vartheta \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix} \tag{5.3.1a}$$

$$\begin{aligned} \begin{pmatrix} \bar{x}^1 \\ \bar{x}^2 \\ \bar{x}^3 \end{pmatrix} &= \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos\vartheta & 0 & \sin\vartheta \\ 0 & 1 & 0 \\ -\sin\vartheta & 0 & \cos\vartheta \end{pmatrix} \\ &\cdot \begin{pmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ \varrho \end{pmatrix} \end{aligned} \tag{5.3.1b}$$

Another set $(r, \vartheta, \varphi, \rho, z, \beta)$ will be called co-rotated cylindrical coordinates and defined as

$$\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos\vartheta & 0 & \sin\vartheta \\ 0 & 1 & 0 \\ -\sin\vartheta & 0 & \cos\vartheta \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix} \tag{5.3.2a}$$

$$\begin{pmatrix} \bar{x}^1 \\ \bar{x}^2 \\ \bar{x}^3 \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos\vartheta & 0 & \sin\vartheta \\ 0 & 1 & 0 \\ -\sin\vartheta & 0 & \cos\vartheta \end{pmatrix} \cdot \begin{pmatrix} \rho\cos\beta \\ \rho\sin\beta \\ z \end{pmatrix} \tag{5.3.2b}$$

They are obviously related to each other by

$$\rho = \varrho\sin\alpha\,, \quad z = \varrho\cos\alpha\,, \tag{5.3.3}$$

Note that in both cases (r, ϑ, φ) are simply the spherical coordinates on M which we had introduced already earlier, and that their induced coordinates on each tangent space are related to those defined above by the coordinate transformation

$$\bar{x}^1 = \bar{r}\sin\vartheta\cos\varphi + \bar{\vartheta}r\cos\vartheta\cos\varphi - \bar{\varphi}r\sin\vartheta\sin\varphi\,, \tag{5.3.4a}$$

$$\bar{x}^2 = \bar{r}\sin\vartheta\sin\varphi + \bar{\vartheta}r\cos\vartheta\sin\varphi + \bar{\varphi}r\sin\vartheta\cos\varphi\,, \tag{5.3.4b}$$

$$\bar{x}^3 = \bar{r}\cos\vartheta - \bar{\vartheta}r\sin\vartheta\,, \tag{5.3.4c}$$

from which follows

$$\begin{pmatrix} \bar{\vartheta}r \\ \bar{\varphi}r\sin\vartheta \\ \bar{r} \end{pmatrix} = \begin{pmatrix} \varrho\sin\alpha\cos\beta \\ \varrho\sin\alpha\sin\beta \\ \varrho\cos\alpha \end{pmatrix} = \begin{pmatrix} \rho\cos\beta \\ \rho\sin\beta \\ z \end{pmatrix}. \tag{5.3.5}$$

The reason for choosing these coordinates becomes clear when one uses them to express the action of the rotation generators (4.1.1) on the tangent bundle. One finds that in either of the two sets of coordinates on TM their canonical lifts have the same form

$$\hat{R}_1 = \sin\varphi\partial_\vartheta + \frac{\cos\varphi}{\tan\vartheta}\partial_\varphi - \frac{\cos\varphi}{\sin\vartheta}\partial_\beta\,, \tag{5.3.6a}$$

$$\hat{R}_2 = -\cos\varphi\partial_\vartheta + \frac{\sin\varphi}{\tan\vartheta}\partial_\varphi - \frac{\sin\varphi}{\sin\vartheta}\partial_\beta\,, \tag{5.3.6b}$$

$$\hat{R}_3 = -\partial_\varphi\,. \tag{5.3.6c}$$

This is due to the fact that they involve only the coordinates $(\vartheta, \varphi, \beta)$, which are common to both coordinate systems. Note that these three coordinates simply denote coordinates have a simple geometric interpretation. The angular coordinates (ϑ, φ) are coordinates on a sphere of constant radius r. A tangent vector to this sphere has the form $\bar{\vartheta}\partial_\vartheta + \bar{\varphi}\partial_\varphi$, and is thus characterized by the equivalent conditions that $\bar{r}$, z or $\cos\alpha$ vanish. The coordinate β determines the orientation of this tangent vector on the sphere, while the remaining coordinate ρ or ϱ determines its length. The fact that only ∂_β appears in the complete lift shows that a rotation of the sphere preserves tangency condition $\bar{r} = 0$ and only changes the orientation of such tangent vectors, but not their length. Another, more physical, but equivalent, interpretation is obtained by realizing that $(\vartheta, \varphi, \beta)$ yield a parametrization of SO(3) in terms of Euler angles, and so they can be interpreted as parametrizing the orientation of a rigid rotor, where (ϑ, φ) indicate the orientation of its main axis of inertia, while β denotes rotation around this main axis. This analogy becomes even more apparent if one defines the vector fields

$$\hat{B}_1 = \sin\beta\partial_\vartheta + \frac{\cos\beta}{\tan\vartheta}\partial_\beta - \frac{\cos\beta}{\sin\vartheta}\partial_\varphi\,, \tag{5.3.7a}$$

$$\hat{B}_2 = -\cos\beta\partial_\vartheta + \frac{\sin\beta}{\tan\vartheta}\partial_\beta - \frac{\sin\beta}{\sin\vartheta}\partial_\varphi\,, \tag{5.3.7b}$$

$$\hat{B}_3 = -\partial_\beta \tag{5.3.7c}$$

on TM, which follow from exchanging the angles φ and β. This can be seen by defining their actions on functions f on TM as

$$\mathcal{R}_j f = i\pounds_{\hat{R}_j} f\,, \quad \mathcal{B}_j f = i\pounds_{\hat{B}_j} f\,. \tag{5.3.8}$$

It then follows that they satisfy the Lie algebra relations

$$[\mathcal{R}_j, \mathcal{R}_k] = i\epsilon_{jkl}\mathcal{R}_l\,, \quad [\mathcal{B}_j, \mathcal{B}_k] = i\epsilon_{jkl}\mathcal{B}_l\,, \quad [\mathcal{B}_j, \mathcal{R}_k] = 0\,, \tag{5.3.9}$$

which correspond to the quantum mechanical operators of space-fixed and body-fixed angular momentum of a rigid rotor. Further defining the operators

$$\mathcal{R}_\pm = \mathcal{R}_1 \pm i\mathcal{R}_2\,, \qquad \mathcal{R}_z = \mathcal{R}_3\,, \qquad \mathcal{R}^2 = \mathcal{R}_1^2 + \mathcal{R}_2^2 + \mathcal{R}_3^2\,, \tag{5.3.10a}$$

$$\mathcal{B}_\pm = \mathcal{B}_1 \pm i\mathcal{B}_2\,, \qquad \mathcal{B}_z = \mathcal{B}_3\,, \qquad \mathcal{B}^2 = \mathcal{B}_1^2 + \mathcal{B}_2^2 + \mathcal{B}_3^2\,, \tag{5.3.10b}$$

one finds that they satisfy the relations

$$[\mathcal{R}_z, \mathcal{R}_\pm] = \pm\mathcal{R}_\pm\,, \qquad [\mathcal{R}_+, \mathcal{R}_-] = 2\mathcal{R}_z\,, \qquad [\mathcal{R}_\pm, \mathcal{R}^2] = [\mathcal{R}_z, \mathcal{R}^2] = 0\,, \tag{5.3.11a}$$

$$[\mathcal{B}_z, \mathcal{B}_\pm] = \pm\mathcal{B}_\pm\,, \qquad [\mathcal{B}_+, \mathcal{B}_-] = 2\mathcal{B}_z\,, \qquad [\mathcal{B}_\pm, \mathcal{B}^2] = [\mathcal{B}_z, \mathcal{B}^2] = 0\,, \tag{5.3.11b}$$

as well as $\mathcal{R}^2 = \mathcal{B}^2$ being the Casimir operator of the rigid rotor. Hence, harmonic functions and tensors on the tangent bundle will arise from representations of the algebra of the rigid rotor.

5.3.2 Harmonic tangent bundle functions

Using the properties of the rotation algebra discussed in the previous section, it is now possible to find a complete set of orthogonal functions on the orbits of the rotation group in the tangent bundle, which are parametrized by the angular coordinates $(\vartheta, \varphi, \beta)$. Such functions can be found by making a separation ansatz of the form

$$f(x, y) = \tilde{f}(r, \varrho, \alpha)Y(\vartheta, \varphi, \beta) = \check{f}(r, \rho, z)Y(\vartheta, \varphi, \beta)\,, \tag{5.3.12}$$

and to realize that the rotation operators introduced earlier act only on the angular part $Y(\vartheta, \varphi, \beta)$. Further, one uses the fact that the three operators $\mathcal{R}^2, \mathcal{R}_z, \mathcal{B}_z$ mutually commute, and so one can find a set of common eigenfunctions. Decomposing the resulting eigenvalue equations with another separation ansatz, and solving the resulting differential equations, taking into account the periodicity of the angular coordinates, one finds that the eigenfunctions form a discrete series of the form

$$\begin{aligned}\mathcal{Y}_{l,m,n}(\vartheta, \varphi, \beta) = {} & N_{l,m,n} e^{im\varphi} e^{in\beta} \cos^{m+n}\frac{\vartheta}{2} \sin^{|m-n|}\frac{\vartheta}{2} \\ & \cdot {}_2F_1\left(\max(m, n) - l, \max(m, n) + l + 1; |m - n| + 1; \sin^2\frac{\vartheta}{2}\right)\,,\end{aligned} \tag{5.3.13}$$

where the normalization constants are given by

$$N_{l,m,n} = (-1)^{\max(m,n)} \frac{\sqrt{(2l+1)}}{|m-n|!} \sqrt{\frac{(l - \min(m, n))!(l + \max(m, n))!}{(l - \max(m, n))!(l + \min(m, n))!}}\,, \tag{5.3.14}$$

and the three parameters must satisfy the conditions

$$l \in \mathbb{N}\,, \quad m, n \in \{-l, -l+1, \ldots, l\} \tag{5.3.15}$$

in order to obtain a well-defined and smooth solution on the orbit of the rotation group. Given the explicit formula, it is straightforward to check that these functions indeed satisfy the eigenvalue equations

$$\mathcal{R}^2\mathcal{Y}_{l,m,n} = l(l+1)\mathcal{Y}_{l,m,n}\,, \quad \mathcal{R}_z\mathcal{Y}_{l,m,n} = m\mathcal{Y}_{l,m,n}\,, \quad \mathcal{B}_z\mathcal{Y}_{l,m,n} = n\mathcal{Y}_{l,m,n}\,. \tag{5.3.16}$$

Further, one finds that functions with identical l can be transformed into each other by application of the ladder operators

$$\mathcal{R}_\pm \mathcal{Y}_{l,m,n} = \sqrt{(l \mp m)(l \pm m + 1)}\mathcal{Y}_{l,m\pm1,n}\,, \tag{5.3.17a}$$

$$\mathcal{B}_\pm \mathcal{Y}_{l,m,n} = \sqrt{(l \mp n)(l \pm n + 1)}\mathcal{Y}_{l,m,n\pm1}\,. \tag{5.3.17b}$$

Finally, we remark that the normalization of these functions is chosen such that they satisfy the relations

$$\int_0^{2\pi}\int_0^{2\pi}\int_0^{\pi} \mathcal{Y}_{l,m,n}(\vartheta,\varphi,\beta)\overline{\mathcal{Y}_{l',m',n'}}(\vartheta,\varphi,\beta)\sin\vartheta\,\mathrm{d}\vartheta\,\mathrm{d}\varphi\,\mathrm{d}\beta = 8\pi^2\delta_{ll'}\delta_{mm'}\delta_{nn'} \tag{5.3.18}$$

of an orthonormal basis, where the overline denotes complex conjugation.

5.3.3 Harmonic d-tensors

Following a similar idea as for the scalar functions on the tangent bundle discussed in the previous section, one can also proceed to study the transformation of d-tensors under the rotation group, and decompose them into irreducible representations. However, there is an important difference which must be taken into account. As explained in section 3.1.3, despite being fiber bundles over the tangent bundle TM, the d-tensor bundles are not natural bundles over TM, since they do not allow for a functorial lift of general diffeomorphisms of TM, but only of those diffeomorphisms which are lifted from the base manifold M. Hence, only the operators $\mathcal{R}_j$ induced by rotations of the base manifold can be extended to act on d-tensors, while the co-rotation operators $\mathcal{B}_j$ cannot. It is thus not possible to work with the extended algebra of the rigid rotor, but one is limited to the common rotation algebra generated by the vector fields R_j.

There are different possibilities to construct harmonic d-tensors. In our work [H7], we followed an approach to construct tensor harmonics on $\mathbb{R}^3$ [107]. This approach starts from the Cartesian coordinate basis $(\partial_1, \partial_2, \partial_3)$ of each tangent space T_xM, and the corresponding dual basis $(\mathrm{d}x^1, \mathrm{d}x^2, \mathrm{d}x^3)$ of T_x^*M. Via the pullback along the tangent bundle projection $\tau : TM \to M$, they induces bases of the pullback bundles τ^*TM and τ^*T^*M, from which the d-tensor bundles are constructed as mentioned in section 2.2.4, and which we denote by the same symbols here for brevity. From these Cartesian coordinate bases, one then constructs the bases

$$\mathbf{e}_0 = \partial_3\,, \quad \mathbf{e}_1 = -\frac{\partial_1 + i\partial_2}{\sqrt{2}}\,, \quad \mathbf{e}_{-1} = \frac{\partial_1 - i\partial_2}{\sqrt{2}}\,, \tag{5.3.19}$$

as well as

$$\mathbf{e}^0 = \mathrm{d}x^3\,, \quad \mathbf{e}^1 = -\frac{\mathrm{d}x^1 + i\mathrm{d}x^2}{\sqrt{2}}\,, \quad \mathbf{e}^{-1} = \frac{\mathrm{d}x^1 - i\mathrm{d}x^2}{\sqrt{2}}\,. \tag{5.3.20}$$

One finds that these bases transform under the fundamental, three-dimensional representation of the rotation group. Hence, they satisfy the relations

$$\mathcal{R}^2\mathbf{e}_m = 2\mathbf{e}_m\,, \quad \mathcal{R}_z\mathbf{e}_m = m\mathbf{e}_m\,, \quad \mathcal{R}_\pm\mathbf{e}_m = \sqrt{(1 \mp m)(2 \pm m)}\mathbf{e}_{m\pm1}\,, \tag{5.3.21}$$

and analogously for the dual basis. One then uses these basis elements in order to construct the harmonic d-tensors via a simple recursion formula. The starting point of this formula are the zeroth-rank d-tensors, which are identified with the scalar tangent bundle spherical harmonics via

$$\overset{m}{\underset{n}{\mathbf{Y}}}{}_l = \mathcal{Y}_{l,m,n}\,. \tag{5.3.22}$$

Here we have centered the index l, as a means to indicate that this object is a scalar function, which transforms neither covariantly nor contravariantly under coordinate transformations, but trivially. This notation is to be contrasted with that of the vector fields

$$\underset{n}{\overset{m}{\mathbf{Y}}}{}_{l'l} = (-1)^{l-m}\sqrt{2l+1}\sum_{m',\mu}\begin{pmatrix} l & l' & 1 \\ m & -m' & -\mu \end{pmatrix}\mathcal{Y}_{l',m',n}\mathbf{e}_{\mu}\,, \tag{5.3.23}$$

where now the lower position of the last index reflects that of the basis element. Conversely, the covectors

$$\underset{n}{\overset{m}{\mathbf{Y}}}{}_{l'}{}^{l} = (-1)^{l-m}\sqrt{2l+1}\sum_{m',\mu}\begin{pmatrix} l & l' & 1 \\ m & -m' & -\mu \end{pmatrix}\mathcal{Y}_{l',m',n}\mathbf{e}^{\mu}\,, \tag{5.3.24}$$

carry an upper index. The same principle is applied also to higher rank tensors, which are recursively defined using the tensor product

$$\underset{n}{\overset{m}{\mathbf{Y}}}{}_{l_0 l_1\cdots l_k} = (-1)^{l_k-m}\sqrt{2l_k+1}\sum_{m',\mu}\begin{pmatrix} l_k & l_{k-1} & 1 \\ m & -m' & -\mu \end{pmatrix}\underset{n}{\overset{m'}{\mathbf{Y}}}{}_{l_0 l_1\cdots l_{k-1}}\otimes\mathbf{e}_{\mu} \tag{5.3.25}$$

with further basis elements, and analogously with covariant and mixed tensors. Here the parentheses denote Clebsch-Gordan coefficients; note that these are non-vanishing if and only of the indices satisfy the conditions

$$l_0 = 0,1,\ldots\,,\quad l_i = |l_{i-1}-1|,\ldots,l_{i-1}+1\,,\quad m = -l_k,\ldots,l_k\,,\quad n = -l_0,\ldots,l_0\,. \tag{5.3.26}$$

Explicitly iterating the recursion formula, one can also write them as the multiple tensor product

$$\underset{n}{\overset{m_k}{\mathbf{Y}}}{}_{l_0 l_1\cdots l_k} = \sum_{\substack{m_0,\ldots,m_{k-1}\\ \mu_1,\ldots,\mu_k}}\mathcal{Y}_{l_0,m_0,n}\mathbf{e}_{\mu_1}\otimes\ldots\otimes\mathbf{e}_{\mu_k} \cdot\prod_{i=1}^{k}(-1)^{l_i-m_i}\sqrt{2l_i+1}\begin{pmatrix} l_i & l_{i-1} & 1 \\ m_i & -m_{i-1} & -\mu_i \end{pmatrix}\,. \tag{5.3.27}$$

It is well know that the Clebsch-Gordan coefficients appear in the decomposition of tensor products of representations of the rotation group into sums of irreducible representations. Applying this fact to the recursion formula given above, taking into account the transformation of the tangent bundle spherical harmonics and the basis vectors under the rotation group, one easily finds that the harmonic d-tensors satisfy the transformation rules

$$\begin{gathered}\mathcal{R}^2\underset{n}{\overset{m}{\mathbf{Y}}}{}_{l_0 l_1\cdots l_k} = l_k(l_k+1)\underset{n}{\overset{m}{\mathbf{Y}}}{}_{l_0 l_1\cdots l_k}\,,\quad \mathcal{R}_z\underset{n}{\overset{m}{\mathbf{Y}}}{}_{l_0 l_1\cdots l_k} = m\underset{n}{\overset{m}{\mathbf{Y}}}{}_{l_0 l_1\cdots l_k}\,,\\ \mathcal{R}_{\pm}\underset{n}{\overset{m}{\mathbf{Y}}}{}_{l_0 l_1\cdots l_k} = \sqrt{(l_k\mp m)(l_k\pm m+1)}\underset{n}{\overset{m\pm 1}{\mathbf{Y}}}{}_{l_0 l_1\cdots l_k}\,.\end{gathered} \tag{5.3.28}$$

Hence, we see that the indices $n, l_0, \ldots, l_k$ label the irreducible representations of dimension $2l_k+1$ in this decomposition, while $m = -l_k, \ldots, l_k$ is the magnetic quantum number, which labels the different basis elements within this representation. Finally, we remark that the normalization is chosen such that they satisfy the orthonormality relations

$$\int_0^{2\pi}\int_0^{2\pi}\int_0^{\pi}\left\langle\underset{n}{\overset{m}{\mathbf{Y}}}{}_{l_0 l_1\cdots l_k}, \underset{n'}{\overset{m'}{\mathbf{Y}}}{}^{l'_0 l'_1\cdots l'_k}\right\rangle(\vartheta,\varphi,\beta)\sin\vartheta\,\mathrm{d}\vartheta\,\mathrm{d}\varphi\,\mathrm{d}\beta = 8\pi^2\delta_{mm'}\delta_{nn'}\prod_{i=0}^{k}\delta_{l_i l'_i}\,, \tag{5.3.29}$$

where the scalar product is defined as

$$\langle A, B\rangle = A^*_{a_1\cdots a_k} B^{a_k\cdots a_1}\,, \tag{5.3.30}$$

and where the star denotes complex conjugation. In our work [H7], we derived further formulas for the transpose, contraction and products of harmonic d-tensors.

5.3.4 Intrinsic approach

In our work [H7], we followed the approach outlined in [107] which can be regarded extrinsic in the sense that it considers harmonic d-tensors on a three-dimensional spaces acted upon by the rotation group. However, as argued in section 5.3.1, the rotation generators act non-trivially only on the angular coordinates and the orientation of tangent vector to the spheres which are the orbits of the rotation group, while leaving any radial coordinate, radial tangent vector component and length of tangent vectors unchanged. One may therefore consider an alternative, intrinsic approach instead, which makes use of d-tensors on the sphere S^2 only, and which we briefly explain in this section. The advantage of this approach is that it makes a proper split of d-tensors into angular and non-angular components, and then expands only the former into harmonics, without imposing any conditions on the structure of the latter.

We now make use of the fact that by restriction to $\bar{r} = 0$, or equivalently $z = 0$ or $\cos\alpha = 0$, the coordinates given in section 5.3.1 describe those tangent vectors with are tangent to the spheres of constant radial coordinate r. It then follows that the tangent bundle TS^2 can be parametrized in terms of the coordinates $(\vartheta, \varphi, \bar{\vartheta}, \bar{\varphi})$, or equivalently $(\vartheta, \varphi, \beta, u)$, where $u = \rho/r = \varrho/r$ measures the length of tangent vectors, and the other coordinates retain their meaning. Recalling from section 5.3.2 that the scalar harmonics $\mathcal{Y}_{l,m,n}$ depend only on the coordinates $(\vartheta, \varphi, \beta)$, one thus sees that they are naturally given as functions on TS^2.

To construct the harmonic d-tensors, one may they proceed in analogy to the construction of tensor harmonics on S^2, by applying suitable linear operators to the scalar harmonics [149]. These are defined due to the fact that the sphere S^2 is canonically equipped with a unique (up to a constant scalar factor) spherically symmetric Finsler geometry, derived from the Finsler function

$$F(\vartheta, \varphi, \beta, u) = u = \sqrt{\bar{\vartheta}^2 + \sin^2\vartheta\,\bar{\varphi}^2}\,. \tag{5.3.31}$$

This Finsler function happens to originate from the canonical Riemannian metric

$$\boldsymbol{\gamma} = \mathrm{d}\vartheta \otimes \mathrm{d}\vartheta + \sin^2\vartheta\,\mathrm{d}\varphi \otimes \mathrm{d}\varphi \tag{5.3.32}$$

on the sphere, which further defines a volume form

$$\boldsymbol{\epsilon} = \sin\vartheta(\mathrm{d}\vartheta \otimes \mathrm{d}\varphi - \mathrm{d}\varphi \otimes \mathrm{d}\vartheta)\,. \tag{5.3.33}$$

We will not enter the details of this construction here, since it depends on the rank of the constructed d-tensors, and restrict ourselves to showing the relation between the intrinsic d-tensors with the extrinsic d-tensors we constructed in the previous section, where we restrict ourselves to covariant tensors for simplicity. For this purpose it is helpful to express the basis elements $\mathbf{e}^m = \underset{0}{\overset{m}{\mathbf{Y}}}0^1$ in spherical coordinates, which yields

$$\mathbf{e}^0 = \cos\vartheta\,\mathrm{d}r - r\sin\vartheta\,\mathrm{d}\vartheta\,, \quad \mathbf{e}^{\pm 1} = -\frac{e^{\pm i\varphi}}{\sqrt{2}}\left(\pm\sin\vartheta\,\mathrm{d}r \pm r\cos\vartheta\,\mathrm{d}\vartheta + ir\sin\vartheta\,\mathrm{d}\varphi\right)\,. \tag{5.3.34}$$

Note in particular that the radial coordinate basis covector is given by the linear combination

$$-\overset{0}{\underset{0}{\mathbf{Y}}}{}_1{}^0 = \frac{\mathcal{Y}_{1,0,0}\mathbf{e}^0 - \mathcal{Y}_{1,-1,0}\mathbf{e}^1 - \mathcal{Y}_{1,1,0}\mathbf{e}^{-1}}{\sqrt{3}} = \mathrm{d}r\,. \tag{5.3.35}$$

By subtracting a suitable multiple of this element, one can split the basis elements in the form

$$\tilde{\mathbf{e}}^m = \tilde{\boldsymbol{\pi}} \cdot \mathbf{e}^m = \frac{2\overset{m}{\underset{0}{\mathbf{Y}}}{}_0{}^1 + \sqrt{2}\overset{m}{\underset{2}{\mathbf{Y}}}{}_0{}^1}{3} = \overset{m}{\underset{0}{\mathbf{Y}}}{}_0{}^{\tilde{1}}\,, \quad \hat{\mathbf{e}}^m = \hat{\boldsymbol{\pi}} \cdot \mathbf{e}^m = \frac{\overset{m}{\underset{0}{\mathbf{Y}}}{}_0{}^1 - \sqrt{2}\overset{m}{\underset{2}{\mathbf{Y}}}{}_0{}^1}{3} = \overset{m}{\underset{0}{\mathbf{Y}}}{}_0{}^{\hat{1}}\,, \tag{5.3.36}$$

where we used a tilde to denote purely angular components and a hat to denote purely radial components. By applying the projectors

$$\tilde{\boldsymbol{\pi}} = -\frac{2\overset{0}{\underset{0}{\mathbf{Y}}}{}_0{}^1{}_0 + \sqrt{2}\overset{0}{\underset{0}{\mathbf{Y}}}{}_2{}^1{}_0}{\sqrt{3}} = \mathrm{d}\vartheta\otimes\partial_\vartheta + \mathrm{d}\varphi\otimes\partial_\varphi\,, \quad \hat{\boldsymbol{\pi}} = -\frac{\overset{0}{\underset{0}{\mathbf{Y}}}{}_0{}^1{}_0 - \sqrt{2}\overset{0}{\underset{0}{\mathbf{Y}}}{}_2{}^1{}_0}{\sqrt{3}} = \mathrm{d}r\otimes\partial_r \tag{5.3.37}$$

to every index of a harmonic d-tensor, one can thus achieve a full decomposition into radial and angular components. Further, using the fact that the canonical d-tensors on the sphere are given by

$$\boldsymbol{\gamma} = -\frac{2\overset{0}{\underset{0}{\mathbf{Y}}}{}_0{}^{10} + \sqrt{2}\overset{0}{\underset{0}{\mathbf{Y}}}{}_2{}^{10}}{\sqrt{3}r^2}\,, \quad \boldsymbol{\epsilon} = -\frac{i\sqrt{6}}{r^2}\overset{0}{\underset{0}{\mathbf{Y}}}{}_{10} \cdot \overset{0}{\underset{0}{\mathbf{Y}}}{}_0{}^{110} = \frac{i\sqrt{2}}{r^2}\overset{0}{\underset{0}{\mathbf{Y}}}{}_1{}^{10}\,, \tag{5.3.38}$$

and decomposing the derivative operators acting on d-tensors on the sphere which arise from this background geometry using the rules for the d-tensors presented in section 5.3.3, one can construct a full d-tensor generalization of the harmonic tensors [149].

6 Transforming teleparallel gravity theories with field space symmetries

In the previous sections we have discussed of transformation groups on the base space of a natural bundle, and the induced transformation on the fields defined on this bundle, as described in section 2.3.1, in the context of gravity theories. In this context, the base space is interpreted as spacetime, and the induced field transformations are interpreted as the change of the mathematical representation of the gravitational field under a change of spacetime coordinates. In this final section, we consider a different type of transformations, which act on the fibers of the field space instead, as laid out in section 2.3.3, and can be interpreted as a transformation of the dynamical variables which describe the gravitational field. The theories we consider here are teleparallel gravity theories, whose underlying geometry we discussed in section 2.2.3, to which we couple a single or multiple scalar fields. We study two different classes of transformations acting on the dynamical fields: conformal transformations in section 6.1 and disformal transformations in section 6.2.

6.1 Conformal transformations

In our work [H8], we studied the behavior of a class of scalar-torsion theories of gravity under conformal transformations, which we summarize in this section. As an introduction, we briefly review the notion of conformal transformations in Riemannian geometry in section 6.1.1, and explain how it relates to the notion of field space symmetries we defined in section 2.3.3. We then translate this concept to teleparallel geometry in section 6.1.2. In section 6.1.3, we discuss a class of scalar-torsion theories of gravity and show how conformal transformations relate different constituents of this class to each other, and can thus be regarded equivalent up to field redefinitions. The aforementioned considerations are generalized to multiple scalar fields in section 6.1.4. Finally, in section 6.1.5, we display a number of invariant quantities which can be used to characterize each equivalence class of scalar-torsion theories.

6.1.1 Conformal transformations of metric geometry

In the following we will consider conformal transformations of teleparallel geometry in the case that these are defined by a single scalar field. The starting point of this discussion is the conformal transformation of the metric, which takes the form

$$\tilde{g}_{\mu\nu} = \mathfrak{c}(\phi) g_{\mu\nu} \,, \tag{6.1.1}$$

where the conformal factor $\mathfrak{c}(\phi)$ is determined by the value of the scalar field ϕ through a positive function $\mathfrak{c} : \mathbb{R} \to \mathbb{R}^+$. Before transferring this notion into the framework of teleparallel geometry, it is worth elucidating on its geometric foundation. Recall that a (Lorentzian) metric is a section $g : M \to \mathrm{LorMet}(M)$ of the bundle of non-degenerate metrics of Lorentzian signature, while a scalar field $\phi : M \to \mathbb{R}$ can be regarded as a section of the trivial bundle $M \times \mathbb{R}$. Hence, the pair (g, ϕ) constitutes a section of the fibered product $E = \mathrm{LorMet}(M) \times_M (M \times \mathbb{R})$, more concisely written as

$$\begin{array}{rccc} (g,\phi) & : & M & \to & E \\ & & x & \mapsto & (g_{\mu\nu}(x), \phi(x)) \end{array} . \tag{6.1.2}$$

Since the signature of the metric is preserved under multiplication by a positive factor, one may consider a transformed section

$$\begin{array}{rccc} ((\mathfrak{c}\circ\phi)g,\phi) & : & M & \to & E \\ & & x & \mapsto & (\mathfrak{c}(\phi(x))g_{\mu\nu}(x), \phi(x)) \end{array} . \tag{6.1.3}$$

Hence, we may consider conformal transformations as an action of the multiplicative group of positive functions $\mathfrak{c} : \mathbb{R} \to \mathbb{R}^+$ on the space of sections of E. This class of transformations may be further enhanced by also considering a transformation of the scalar field ϕ. Such a field redefinition is defined by a bijective function $f : \mathbb{R} \to \mathbb{R}$, and acts on a section (g, ϕ) to yield

$$\begin{array}{rccc} (g,f\circ\phi) & : & M & \to & E \\ & & x & \mapsto & (g_{\mu\nu}(x), f(\phi(x))) \end{array} . \tag{6.1.4}$$

Combining both operations, a pair $(\mathfrak{c}, f)$ acts on a section (g, ϕ) as

$$\begin{array}{rccc} ((\mathfrak{c}\circ\phi)g,f\circ\phi) & : & M & \to & E \\ & & x & \mapsto & (\mathfrak{c}(\phi(x))g_{\mu\nu}(x), f(\phi(x))) \end{array} . \tag{6.1.5}$$

Hence, it follows that the combined scalar field redefinitions and conformal transformations form a group with group operation

$$(\mathfrak{c}, f) \cdot (\mathfrak{c}', f') = ((\mathfrak{c} \circ f')\mathfrak{c}', f \circ f') \,, \tag{6.1.6}$$

so that the inverse transformation is given by

$$(\mathfrak{c}, f)^{-1} = \left(\frac{1}{\mathfrak{c} \circ f^{-1}}, f^{-1}\right) . \tag{6.1.7}$$

It follows from these considerations, that the class of transformations we consider here can be treated using the framework laid out in section 2.3.3.

6.1.2 Conformal transformations of teleparallel geometry

Having clarified the concise mathematical framework, we can no proceed to conformal transformations, combined with scalar field redefinitions, in teleparallel geometry. We therefore now consider a transformation of the tetrad and scalar field given by

$$\tilde{\theta}^A{}_\mu = \mathfrak{C}(\phi)\theta^A{}_\mu\,, \quad \tilde{\phi} = f(\phi)\,, \tag{6.1.8}$$

with a positive function $\mathfrak{C} : \mathbb{R} \to \mathbb{R}^+$. Note that we do not transform the teleparallel spin connection. This is justified by the fact that any transformation, which retains the flatness and antisymmetry of the spin connection, could be absorbed into a local Lorentz transformation. It then follows that the metric (2.2.22) transforms as

$$\tilde{g}_{\mu\nu} = \mathfrak{C}^2(\phi)g_{\mu\nu}\,, \tag{6.1.9}$$

and so the conformal transformation of the tetrad induces a conformal transformation of the metric of the form (6.1.1) with $\mathfrak{c} = \mathfrak{C}^2$. For further studies of these transformations, however, it turns out to be more convenient to express the conformal transformation as

$$\mathfrak{C}(\phi) = e^{\gamma(\phi)}\,, \tag{6.1.10}$$

in terms of a function $\gamma : \mathbb{R} \to \mathbb{R}$, so that the transformation of the tetrad and metric reads

$$\tilde{g}_{\mu\nu} = e^{2\gamma(\phi)}g_{\mu\nu}\,, \quad \tilde{\theta}^A{}_\mu = e^{\gamma(\phi)}\theta^A{}_\mu\,. \tag{6.1.11}$$

With these definitions in place, it is now straightforward to calculate the transformation of further relevant geometric quantities such as the torsion and contortion tensors. We will not display them here, and defer a discussion to section 6.2.1, where we give a more general expression for disformal transformations. In the following sections, we will focus only on the transformation of terms which directly appear in the action functional of the teleparallel gravity theories we consider.

6.1.3 Conformal transformations of teleparallel gravity actions

In our work [H8] we have considered a class of teleparallel gravity theories, whose action takes the general form

$$S_g\left[\theta^A, \overset{\bullet}{\omega}{}^A{}_B, \phi, \chi^I\right] = S_g\left[\theta^A, \overset{\bullet}{\omega}{}^A{}_B, \phi\right] + S_m\left[\theta^A, \phi, \chi^I\right]\,, \tag{6.1.12}$$

and thus splits into a gravitational part S_g, depending on the tetrad, spin connection and scalar field, and a matter part S_m, depending on the tetrad, scalar field and arbitrary matter fields χ^I. For the gravitational part of the action, we assumed the form

$$S_g\left[\theta^A, \overset{\bullet}{\omega}{}^A{}_B, \phi\right] = \frac{1}{2\kappa^2}\int_M \left[-\mathcal{A}(\phi)\mathbb{T} + 2\mathcal{B}(\phi)X + 2\mathcal{C}(\phi)Y - 2\kappa^2\mathcal{V}(\phi)\right]\theta \mathrm{d}^4x\,, \tag{6.1.13}$$

where the appearing terms are the torsion scalar

$$\mathbb{T} = \frac{1}{4}\overset{\bullet}{T}{}^{\mu\nu\rho}\overset{\bullet}{T}{}_{\mu\nu\rho} + \frac{1}{2}\overset{\bullet}{T}{}^{\mu\nu\rho}\overset{\bullet}{T}{}_{\rho\nu\mu} - \overset{\bullet}{T}{}^\mu{}_{\mu\rho}\overset{\bullet}{T}{}_\nu{}^{\nu\rho}\,, \tag{6.1.14}$$

the scalar field kinetic term

$$X = -\frac{1}{2}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}\,, \tag{6.1.15}$$

as well as the derivative coupling term

$$Y = g^{\mu\nu}\overset{\bullet}{T}^{\rho}{}_{\rho\mu}\phi_{,\nu}\,. \tag{6.1.16}$$

A particular action from this general class is defined by the choice of the functions $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{V}$ of the scalar field. Another function of the scalar field enters the matter action, which we assumed to be of the form

$$S_m\left[\theta^A, \phi, \chi^I\right] = S_m^{\mathfrak{J}}\left[e^{\alpha(\phi)}\theta^A, \chi^I\right]\,, \tag{6.1.17}$$

so that the scalar field enters only through a conformal rescaling of the tetrad.

Under a combined conformal transformation of the tetrad and scalar field redefinition of the form (6.1.8), using the parametrization (6.1.11), we find that the scalar quantities constituting the action transform as

$$\tilde{\mathbb{T}} = e^{-2\gamma(\phi)}\left(\mathbb{T} + 4\gamma'(\phi)Y + 12\gamma'^2(\phi)X\right)\,, \tag{6.1.18a}$$

$$\tilde{Y} = e^{-2\gamma(\phi)}f'(\phi)(Y + 6\gamma'(\phi)X)\,, \tag{6.1.18b}$$

$$\tilde{X} = e^{-2\gamma(\phi)}f'^2(\phi)X\,, \tag{6.1.18c}$$

and so it follows that the transformed quantities can be expressed as a linear combination of the original quantities, with coefficients given by functions of the scalar field. Due to this property, one finds that transforming any teleparallel gravity action of the form (6.1.12), with gravitational part (6.1.13) and matter part (6.1.17), leads to an action which is of the same form. This transformed action can be derived explicitly as follows. Following the framework given in section 2.3.3, we demand that the transformed action functional, evaluated at the transformed fields, takes the same value as evaluating the original action at the original fields. For the class of action and transformations we consider here, we thus pose the condition

$$\tilde{S}_g\left[\tilde{\theta}^A, \overset{\bullet}{\omega}{}^A{}_B, \tilde{\phi}\right] = S_g\left[\theta^A, \overset{\bullet}{\omega}{}^A{}_B, \phi\right]\,, \quad \tilde{S}_m\left[\tilde{\theta}^A, \tilde{\phi}, \chi^I\right] = S_m\left[\theta^A, \phi, \chi^I\right]\,. \tag{6.1.19}$$

The calculation of the transformed actions is straightforward, using the transformation rules (6.1.18) of the constituting terms. For the gravitational action (6.1.13), one finds

$$\begin{aligned}
\tilde{S}_g\left[\tilde{\theta}^A, \overset{\bullet}{\omega}{}^A{}_B, \tilde{\phi}\right] &= \frac{1}{2\kappa^2}\int_M\left[-\tilde{\mathcal{A}}(\tilde{\phi})\tilde{\mathbb{T}} + 2\tilde{\mathcal{B}}(\tilde{\phi})\tilde{X} + 2\tilde{\mathcal{C}}(\tilde{\phi})\tilde{Y} - 2\kappa^2\tilde{\mathcal{V}}(\tilde{\phi})\right]\tilde{\theta}\mathrm{d}^4x \\
&= \frac{1}{2\kappa^2}\int_M\Big\{-e^{2\gamma(\phi)}\tilde{\mathcal{A}}(f(\phi))\mathbb{T} - 2\kappa^2e^{4\gamma(\phi)}\tilde{\mathcal{V}}(f(\phi)) \\
&\quad + 2e^{2\gamma(\phi)}\left[\tilde{\mathcal{B}}(f(\phi))f'^2(\phi) - 6\tilde{\mathcal{A}}(f(\phi))\gamma'^2(\phi) + 6\tilde{\mathcal{C}}(f(\phi))f'(\phi)\gamma'(\phi)\right]X \\
&\quad + 2e^{2\gamma(\phi)}\left[\tilde{\mathcal{C}}(f(\phi))f'(\phi) - 2\tilde{\mathcal{A}}\gamma'(\phi)\right]Y\Big\}\theta\mathrm{d}^4x\,,
\end{aligned} \tag{6.1.20}$$

while the matter action obeys the transformation

$$\tilde{S}_m\left[\tilde{\theta}^A, \tilde{\phi}, \chi^I\right] = S_m^{\mathfrak{J}}\left[e^{\tilde{\alpha}(\tilde{\phi})}\tilde{\theta}^A, \chi^I\right] = S_m^{\mathfrak{J}}\left[e^{\tilde{\alpha}(f(\phi))+\gamma(\phi)}\theta^A, \chi^I\right]\,. \tag{6.1.21}$$

It thus follows that the transformation we consider indeed retains the form of the action, and that the parameter functions defining a particular action from the general class transform as

$$\mathcal{A} = e^{2\gamma}\tilde{\mathcal{A}}\,, \tag{6.1.22a}$$

$$\mathcal{B} = e^{2\gamma}\left(\tilde{\mathcal{B}}f'^2 - 6\tilde{\mathcal{A}}\gamma'^2 + 6\tilde{\mathcal{C}}f'\gamma'\right), \tag{6.1.22b}$$
$$\mathcal{C} = e^{2\gamma}\left(\tilde{\mathcal{C}}f' - 2\tilde{\mathcal{A}}\gamma'\right), \tag{6.1.22c}$$
$$\mathcal{V} = e^{4\gamma}\tilde{\mathcal{V}}, \tag{6.1.22d}$$
$$\alpha = \tilde{\alpha} + \gamma, \tag{6.1.22e}$$

which is reminiscent of a similar set of transformation rules for scalar-curvature theories of gravity [37]. Here we omitted the function arguments for brevity; it is to be understood that the transformed functions (carrying a tilde) are evaluated at $f(\phi)$, while all other functions, including γ and f, are evaluated at ϕ.

6.1.4 Generalization to multiple scalar fields

The considerations given above are easily generalized to the case of multiple scalar fields, in a similar way as scalar-curvature theories of gravity can be generalized to multi-scalar-curvature theories [27]. As argued in section 2.2.5, one may regard the space of values of a scalar field multiplet $\boldsymbol{\phi}$ as a manifold F, which constitutes the fibers of a trivial bundle $M \times F$, and the individual scalar fields $\phi^{\mathfrak{a}}$ as the corresponding coordinate expressions obtained by choosing coordinates on F. With this geometric interpretation, it is straightforward to generalize the class of transformations discussed in section 6.1.2 to multiple scalar fields, by making the necessary substitutions. It follows that conformal transformations are parametrized by positive functions $\mathfrak{C} : F \to \mathbb{R}^+$, or with $\mathfrak{C} = e^{\gamma}$ equivalently through $\gamma : F \to \mathbb{R}$. Scalar field redefinitions are induced by a diffeomorphism $\boldsymbol{f} : F \to F$, which can also be interpreted as a change of coordinates on the scalar field space. Together, they induce the transformation

$$\tilde{\theta}^A{}_\mu = \mathfrak{C}(\boldsymbol{\phi})\theta^A{}_\mu\,, \quad \tilde{\phi}^{\mathfrak{a}} = f^{\mathfrak{a}}(\boldsymbol{\phi})\,, \tag{6.1.23}$$

which is a direct generalization of the transformation (6.1.8). Using the geometric interpretation given at the beginning of this section, it is also straightforward to generalize the class of scalar-torsion gravity theories to multiple scalar fields. We start by retaining the split

$$S_g\left[\theta^A, \overset{\bullet}{\omega}{}^A{}_B, \phi^{\mathfrak{a}}, \chi^I\right] = S_g\left[\theta^A, \overset{\bullet}{\omega}{}^A{}_B, \phi^{\mathfrak{a}}\right] + S_m\left[\theta^A, \phi^{\mathfrak{a}}, \chi^I\right] \tag{6.1.24}$$

of the action into a gravitational and matter part, simply replacing the single scalar field ϕ by a scalar field multiplet. Next, the gravitational part (6.1.13) becomes

$$S_g\left[\theta^A, \overset{\bullet}{\omega}{}^A{}_B, \phi^{\mathfrak{a}}\right] = \frac{1}{2\kappa^2}\int_M \left[-\mathcal{A}(\boldsymbol{\phi})\mathbb{T} + 2\mathcal{B}_{\mathfrak{ab}}(\boldsymbol{\phi})X^{\mathfrak{ab}} + 2\mathcal{C}_{\mathfrak{a}}(\boldsymbol{\phi})Y^{\mathfrak{a}} - 2\kappa^2\mathcal{V}(\boldsymbol{\phi})\right]\theta \mathrm{d}^4x\,. \tag{6.1.25}$$

Here the scalar field kinetic term (6.1.15) and kinetic coupling term (6.1.16) have been generalized to the expressions

$$X^{\mathfrak{ab}} = -\frac{1}{2}g^{\mu\nu}\phi^{\mathfrak{a}}_{,\mu}\phi^{\mathfrak{b}}_{,\nu}\,, \quad Y^{\mathfrak{a}} = \overset{\bullet}{T}_\mu{}^{\mu\nu}\phi^{\mathfrak{a}}_{,\nu}\,. \tag{6.1.26}$$

Note that the former is symmetric in its two indices; hence, the same restriction applies to the parameter function $\mathcal{B}_{\mathfrak{ab}}$ in the action. Further, all parameter functions now depend on the whole scalar field multiplet $\boldsymbol{\phi}$. This also applies to the generalized matter action

$$S_m\left[\theta^A, \phi^{\mathfrak{a}}, \chi^I\right] = S_m^{\mathfrak{J}}\left[e^{\alpha(\boldsymbol{\phi})}\theta^A, \chi^I\right], \tag{6.1.27}$$

which differs from the single-field case by the generalized dependence of the parameter function α on all scalar fields. Applying the transformation (6.1.23), one finds that the terms in the action (6.1.25) transform as

$$\tilde{\mathbb{T}} = e^{-2\gamma(\boldsymbol{\phi})}\left(\mathbb{T} + 4\gamma_{,\mathfrak{a}}(\boldsymbol{\phi})Y^{\mathfrak{a}} + 12\gamma_{,\mathfrak{a}}(\boldsymbol{\phi})\gamma_{,\mathfrak{b}}(\boldsymbol{\phi})X^{\mathfrak{ab}}\right), \tag{6.1.28a}$$

$$\tilde{Y}^{\mathfrak{a}} = e^{-2\gamma(\boldsymbol{\phi})}f^{\mathfrak{a}}_{,\mathfrak{b}}(\boldsymbol{\phi})\left(Y^{\mathfrak{b}} + 6\gamma_{,\mathfrak{c}}(\boldsymbol{\phi})X^{\mathfrak{bc}}\right), \tag{6.1.28b}$$

$$\tilde{X}^{\mathfrak{ab}} = e^{-2\gamma(\boldsymbol{\phi})}f^{\mathfrak{a}}_{,\mathfrak{c}}(\boldsymbol{\phi})f^{\mathfrak{b}}_{,\mathfrak{d}}(\boldsymbol{\phi})X^{\mathfrak{cd}}, \tag{6.1.28c}$$

where we note the appearance of the gradient $\gamma_{,\mathfrak{a}}$ of γ and the Jacobian $f^{\mathfrak{a}}_{,\mathfrak{b}}$ of $\boldsymbol{f}$, replacing their simple derivatives. Finally, applying this transformation to the action shows that also the class of multi-scalar-torsion theories we consider in this section retains its form under conformal transformations, with the transformation of the parameter functions given by

$$\mathcal{A} = e^{2\gamma}\tilde{\mathcal{A}}, \tag{6.1.29a}$$

$$\mathcal{B}_{\mathfrak{ab}} = e^{2\gamma}\left(\tilde{\mathcal{B}}_{\mathfrak{cd}}f^{\mathfrak{c}}_{,\mathfrak{a}}f^{\mathfrak{d}}_{,\mathfrak{b}} - 6\tilde{\mathcal{A}}\gamma_{,\mathfrak{a}}\gamma_{,\mathfrak{b}} + 6\tilde{\mathcal{C}}_{\mathfrak{c}}f^{\mathfrak{c}}_{,(\mathfrak{a}}\gamma_{,\mathfrak{b})}\right), \tag{6.1.29b}$$

$$\mathcal{C}_{\mathfrak{a}} = e^{2\gamma}\left(\tilde{\mathcal{C}}_{\mathfrak{b}}f^{\mathfrak{b}}_{,\mathfrak{a}} - 2\tilde{\mathcal{A}}\gamma_{,\mathfrak{a}}\right), \tag{6.1.29c}$$

$$\mathcal{V} = e^{4\gamma}\tilde{\mathcal{V}}, \tag{6.1.29d}$$

$$\alpha = \tilde{\alpha} + \gamma\,. \tag{6.1.29e}$$

Also here we omitted all function arguments, as we did in the single-field case (6.1.22). Here it is to be understood that transformed quantities, carrying a tilde, are evaluated at $\boldsymbol{f}(\boldsymbol{\phi})$, with all other quantities evaluated at $\boldsymbol{\phi}$.

6.1.5 Invariants in scalar-torsion gravity

In the field of scalar-curvature theories of gravity, a highly debated question concerns the invariance or non-invariance of physical quantities under conformal transformations of the metric [24, 35, 28, 26, 143, 34, 23, 146]. An important contribution to answering this question has been obtained by the derivation of a parametrization of scalar-curvature gravity actions through a set of functions which is invariant under conformal transformations [112], showing the equivalence of actions which are related by conformal transformations and scalar field redefinitions. This approach has subsequently been used to express various physical quantities in terms of such invariants [113, 124, 111, 114]. Further, this work has also been generalized to multiple scalar fields [123, 82].

The class of scalar-torsion theories discussed here allows to construct a set of invariant quantities in full analogy to the scalar-curvature case, as we have shown in our work [H8]. Here we skip the single-field case, and proceed immediately with the multi-scalar-torsion case, which is a generalization of the former, and which gives a clearer understanding of the geometric nature of the underlying transformations. Hence, we consider the transformation (6.1.29). From the transformation behavior of $\mathcal{A}$, $\mathcal{V}$ and α immediately follows that the combinations

$$\mathcal{I}_1 = \frac{e^{2\alpha}}{\mathcal{A}}, \quad \mathcal{I}_2 = \frac{\mathcal{V}}{\mathcal{A}^2} \tag{6.1.30}$$

are invariant under conformal transformations, while under a scalar field redefinition they obey the transformation

$$\mathcal{I}_1(\boldsymbol{\phi}) = \tilde{\mathcal{I}}_1(\boldsymbol{f}(\boldsymbol{\phi}))\,, \quad \mathcal{I}_2(\boldsymbol{\phi}) = \tilde{\mathcal{I}}_2(\boldsymbol{f}(\boldsymbol{\phi}))\,, \tag{6.1.31}$$

where we explicitly wrote out the previously omitted arguments of these functions for clarity, as a reminder that they are evaluated at the untransformed / transformed values of the scalar fields, respectively. One sees that these invariants behave as scalar functions $\mathcal{I}_{1,2} : F \to \mathbb{R}$ on the scalar field space F. Similarly, one can construct the quantities

$$\mathcal{F}_{\mathfrak{ab}} = \frac{2\mathcal{A}\mathcal{B}_{\mathfrak{ab}} - 6\mathcal{A}_{,(\mathfrak{a}}\mathcal{C}_{\mathfrak{b})} - 3\mathcal{A}_{,\mathfrak{a}}\mathcal{A}_{,\mathfrak{b}}}{4\mathcal{A}^2}\,, \quad \mathcal{H}_{\mathfrak{a}} = \frac{\mathcal{C}_{\mathfrak{a}} + \mathcal{A}_{,\mathfrak{a}}}{2\mathcal{A}}\,, \tag{6.1.32}$$

which combine the parameter functions $\mathcal{B}_{\mathfrak{ab}}$ and $\mathcal{C}_{\mathfrak{a}}$ with other terms, such that their conformal transformations cancel each other, and they become invariant under conformal transformations. Note that there are also other combinations which serve this purpose, such as the combinations

$$\mathcal{G}_{\mathfrak{ab}} = \frac{\mathcal{B}_{\mathfrak{ab}} - 6\alpha_{,(\mathfrak{a}}\mathcal{C}_{\mathfrak{b})} - 6\alpha_{,\mathfrak{a}}\alpha_{,\mathfrak{b}}\mathcal{A}}{2e^{2\alpha}}\,, \quad \mathcal{K}_{\mathfrak{a}} = \frac{\mathcal{C}_{\mathfrak{a}} + 2\alpha_{,\mathfrak{a}}\mathcal{A}}{2e^{2\alpha}}\,, \tag{6.1.33}$$

which are related to the previously defined ones via

$$\mathcal{F}_{\mathfrak{ab}} = \mathcal{I}_1\mathcal{G}_{\mathfrak{ab}} + 3\mathcal{I}_{1,(\mathfrak{a}}\mathcal{K}_{\mathfrak{b})} - \frac{3\mathcal{I}_{1,\mathfrak{a}}\mathcal{I}_{1,\mathfrak{b}}}{4\mathcal{I}_1^2}\,, \quad \mathcal{H}_{\mathfrak{a}} = \mathcal{I}_1\mathcal{K}_{\mathfrak{a}} - \frac{\mathcal{I}_{1,\mathfrak{a}}}{2\mathcal{I}_1}\,. \tag{6.1.34}$$

Also for these indexed quantities it is straightforward to calculate the behavior under scalar field redefinitions. Again omitting the function arguments, keeping in mind that transformed quantities are evaluated at $\boldsymbol{f}(\boldsymbol{\phi})$, we see that they obey the transformation rules

$$\mathcal{F}_{\mathfrak{ab}} = f^{\mathfrak{c}}_{,\mathfrak{a}}f^{\mathfrak{d}}_{,\mathfrak{b}}\tilde{\mathcal{F}}_{\mathfrak{cd}}\,, \quad \mathcal{H}_{\mathfrak{a}} = f^{\mathfrak{b}}_{,\mathfrak{a}}\tilde{\mathcal{H}}_{\mathfrak{b}}\,, \quad \mathcal{G}_{\mathfrak{ab}} = f^{\mathfrak{c}}_{,\mathfrak{a}}f^{\mathfrak{d}}_{,\mathfrak{b}}\tilde{\mathcal{G}}_{\mathfrak{cd}}\,, \quad \mathcal{K}_{\mathfrak{a}} = f^{\mathfrak{b}}_{,\mathfrak{a}}\tilde{\mathcal{K}}_{\mathfrak{b}}\,. \tag{6.1.35}$$

Note the appearance of the Jacobian $f^{\mathfrak{b}}_{,\mathfrak{a}}$. It follows that they transform as covariant tensors of rank 1 and 2, respectively, on the field space manifold F. This finding provides us a clear geometric interpretation of the functions defining the scalar-torsion gravity action.

6.2 Disformal transformations

In the previous section we have discussed the role of conformal transformations in scalar-torsion theories of gravity. The characteristic feature of such transformations is the fact that the tetrad, which constitutes one of the fundamental fields in teleparallel gravity theories, is rescaled by the transformation, but retains its orientation. This condition is relaxed in the case of disformal transformations, which we discussed in our work [H9], and which we summarize here. We briefly introduce the concept of disformal transformations of the metric and the tetrad in section 6.2.1, and show how it affects the teleparallel geometry. These transformations are used to derive an invariant class of scalar-torsion theories of gravity in section 6.2.2. In order to give a geometric interpretation of these transformations following our discussion in section 2.3.3, we generalize our previous work to multiple scalar fields in section 6.2.3.

6.2.1 Disformal transformations of teleparallel geometry

The class of conformal transformations discussed in the previous section can further be generalized to the notion of *disformal* transformations. Again we first consider the most simple case in which the dynamical fields are a metric $g_{\mu\nu}$ and a single scalar field ϕ. In this case, the most general class of disformal transformations can be written in the form [9]

$$\tilde{g}_{\mu\nu} = \mathfrak{c}(\phi, X)g_{\mu\nu} + \mathfrak{d}(\phi, X)\partial_\mu\phi\partial_\nu\phi\,, \tag{6.2.1}$$

where the two free functions $\mathfrak{c}$ and $\mathfrak{d}$ depend not only on the value ϕ of the scalar field itself, but also on the kinetic term (6.1.15). In order to be a proper transformation of the metric, also the transformed metric must be non-degenerate and of Lorentzian signature. Note that the determinant of the metric transforms as

$$\tilde{g} = \mathfrak{c}^3(\mathfrak{c} - 2X\mathfrak{d})g\,. \tag{6.2.2}$$

Hence, in the following we will assume $\mathfrak{c} > 0$ and $\mathfrak{c} - 2X\mathfrak{d} > 0$.

As with conformal transformations, also disformal transformations of the metric can be obtained from a corresponding set of transformations of the tetrad. These can be defined as [33]

$$\tilde{\theta}^A{}_\mu = \mathfrak{C}(\phi, X)\theta^A{}_\mu + \mathfrak{D}(\phi, X)\eta^{AB}e_B{}^\nu\partial_\mu\phi\partial_\nu\phi = \mathfrak{C}(\phi, X)\theta^A{}_\mu + \mathfrak{D}(\phi, X)g^{\nu\rho}\theta^A{}_\rho\partial_\mu\phi\partial_\nu\phi\,, \tag{6.2.3}$$

with two free functions $\mathfrak{C}$ and $\mathfrak{D}$. One easily checks that this yields a disformal transformation of the form (6.2.1) for the metric; by direct calculation one finds

$$\tilde{g}_{\mu\nu} = \mathfrak{C}^2 g_{\mu\nu} + 2\mathfrak{D}(\mathfrak{C} - X\mathfrak{D})\partial_\mu\phi\partial_\nu\phi\,, \tag{6.2.4}$$

so that the free functions are related by

$$\mathfrak{c} = \mathfrak{C}^2\,, \quad \mathfrak{d} = 2\mathfrak{D}(\mathfrak{C} - X\mathfrak{D})\,. \tag{6.2.5}$$

To obtain conditions on these functions $\mathfrak{C}$ and $\mathfrak{D}$, which guarantee that the transformation of the tetrad, and hence also the metric, is non-degenerate, one may return to the conditions derived for $\mathfrak{c}$ and $\mathfrak{d}$. However, it is simpler to directly calculate the transformation of the determinant of the tetrad, which reads

$$\tilde{\theta} = \mathfrak{C}^3(\mathfrak{C} - 2X\mathfrak{D})\theta\,, \tag{6.2.6}$$

and so one obtains the conditions $\mathfrak{C} > 0$ and $\mathfrak{C} - 2X\mathfrak{D} > 0$. A remarkable fact is the consistency with the previously obtained transformation (6.2.2) of the metric determinant, which follows from the non-trivial relation

$$(\mathfrak{C} - 2X\mathfrak{D})^2 = \mathfrak{C}^2 - 2X\mathfrak{D}(2\mathfrak{C} - 2X\mathfrak{D}) = \mathfrak{c} - 2X\mathfrak{d}\,, \tag{6.2.7}$$

which follows from the relation (6.2.5) between the parameter functions. The positivity conditions also appear in the transformation of the inverse tetrad, which reads

$$\tilde{e}_A{}^\mu = \frac{1}{\mathfrak{C}}\left(e_A{}^\mu - \frac{\mathfrak{D}}{\mathfrak{C} - 2X\mathfrak{D}}g^{\mu\nu}e_A{}^\rho\partial_\nu\phi\partial_\rho\phi\right)\,. \tag{6.2.8}$$

Hence, in the following we will consider only transformations which satisfy these conditions.

In our work [H9], disformal transformations are studied in the context of scalar-torsion theories of gravity. In addition to the tetrad and the scalar field, the underlying teleparallel geometry also depends on a spin connection, whose transformation behavior must be specified. However, the situation is identical to the case of conformal transformations, that any non-trivial transformation of the spin connection could simply be absorbed into a local Lorentz transformation, and so we assume it to be invariant, $\tilde{\omega}^A{}_{B\mu} = \omega^A{}_{B\mu}$. Under this assumption, the transformation of the torsion takes the form

$$\tilde{T}^A{}_{\mu\nu} = \mathfrak{C}T^A{}_{\mu\nu} + 2\partial_{[\mu}\mathfrak{C}\theta^A{}_{\nu]} + 2\eta^{AB}e_B{}^\rho\left(\partial_\rho\phi\partial_{[\mu}\mathfrak{D}\partial_{\nu]}\phi + \mathfrak{D}\partial_{[\nu}\phi\nabla_{\mu]}\partial_\rho\phi\right)\,, \tag{6.2.9}$$

while the contortion transforms according to

$$\tilde{K}_{AB\mu} = K_{AB\mu} + 2e_{[A}{}^{\alpha}e_{B]}{}^{\beta}\left(\frac{\mathfrak{D}\mathring{\nabla}_\mu\mathring{\nabla}_\alpha\phi\mathring{\nabla}_\beta\phi}{\mathfrak{C} - 2X\mathfrak{D}} + \frac{\mathfrak{D}^2\mathring{\nabla}_\mu\phi\mathring{\nabla}_\alpha\phi\mathring{\nabla}_\beta X}{\mathfrak{C}(\mathfrak{C} - 2X\mathfrak{D})} + \frac{\mathfrak{C}_{,\phi}\mathring{\nabla}_\alpha\phi g_{\beta\mu}}{\mathfrak{C} - 2X\mathfrak{D}}\right.$$
$$\left. + \frac{\mathfrak{C}_{,X}\mathfrak{D}g_{\alpha\mu}\mathring{\nabla}_\beta\phi\mathring{\nabla}_\gamma\phi\mathring{\nabla}^\gamma X}{\mathfrak{C}(\mathfrak{C} - 2X\mathfrak{D})} + \frac{\mathfrak{D}_{,X}\mathring{\nabla}_\alpha X\mathring{\nabla}_\beta\phi\mathring{\nabla}_\mu\phi}{\mathfrak{C}} + \frac{\mathfrak{C}_{,X}\mathring{\nabla}_\alpha X g_{\beta\mu}}{\mathfrak{C}}\right) . \quad (6.2.10)$$

Here we wrote the transformations in terms of tensor components [50]; in our work [H9], we made use of the language of differential forms.

6.2.2 Disformal transformations of teleparallel gravity actions

In analogy to the class of teleparallel gravity actions which is invariant under conformal transformations discussed in section 6.1.3, one may aim to construct a class of actions which is invariant under disformal transformations. We have proposed multiple such classes in our work [H9], based on the following principle. Since the dynamics of the gravitational field, encoded in the tetrad and the spin connection, enters into the action through the torsion tensor $T^\mu{}_{\nu\rho}$, it will necessarily appear in any teleparallel gravity action. To obtain a scalar Lagrangian, its indices must be contracted with other tensor fields. To simplify the study of disformal transformations for such scalar Lagrangian, it is helpful to first transform the torsion tensor into the equivalent expression

$$A^{1\,A}{}_{BC} = T^A{}_{BC} = \theta^A{}_\mu e_B{}^\nu e_C{}^\rho T^\mu{}_{\nu\rho} , \quad (6.2.11)$$

which transforms as a tensor under local Lorentz transformations, but as a scalar under spacetime diffeomorphisms. The advantage of this change of index characters is that any number of such Lorentz tensors / spacetime scalars can be combined and contracted with the help of the Minkowski metric η_{AB}, which is invariant under disformal transformations, in contrast to the spacetime metric $g_{\mu\nu}$, which transforms non-trivially. Hence, to obtain the transformation of a teleparallel gravity action, it is sufficient to derive the disformal transformations of the constituting Lorentz tensors / spacetime scalars, in addition to the transformation (6.2.6) of the tetrad determinant, which is required to obtain a Lagrangian density from the scalar Lagrangian. For the torsion tensor (6.2.11), one finds that the disformal transformation can be expressed as a sum of the terms

$$A^{2\,A}{}_{BC} = 2T^{AD}{}_{[C}\phi_{,B]}\phi_{,D} , \quad A^{3\,A}{}_{BC} = 2\phi_{,[B}\delta^A_{C]} , \quad A^{4\,A}{}_{BC} = 2X_{,[B}\delta^A_{C]} ,$$
$$A^{5\,A}{}_{BC} = 2\eta^{AD}X_{,[B}\phi_{,C]}\phi_{,D} , \quad A^{6\,A}{}_{BC} = 2\phi_{,[C}\pi^A{}_{B]} , \quad A^{7\,A}{}_{BC} = 2\eta^{DE}\phi_{,D}X_{,E}\phi_{,[B}\delta^a_{C]} . \quad (6.2.12)$$

with coefficients which are functions of ϕ and X. Further, it turns out that all these terms share the same transformation behavior, which can thus be expressed in the common form

$$\tilde{A}^{I\,A}{}_{BC} = \sum_{J=1}^{7} M^I{}_J(\phi, X)A^{J\,A}{}_{BC} , \quad (6.2.13)$$

with a matrix $M^I{}_J$ of coefficient functions; we do not list them here for brevity - a full list is given in [72].

From the transformation behavior of the terms $A^{I\,A}{}_{BC}$ follows that any gravitational action of the form

$$S_g\left[\theta^A, \overset{\bullet}{\omega}{}^A{}_B, \phi\right] = \int_M \left[\sum_N \sum_{\{I_i\}} H_{I_1\cdots I_N\,A_1\cdots A_N}{}^{B_1\cdots B_N C_1\cdots C_N} A^{I_1 A_1}{}_{B_1 C_1} \cdots A^{I_N A_N}{}_{B_N C_N}\right] \theta\, \mathrm{d}^4x\,, \tag{6.2.14}$$

retains this general form under the action of a disformal transformation. Here the expressions $H_{I_1\cdots I_N\,A_1\cdots A_N}{}^{B_1\cdots B_N C_1\cdots C_N}$ are tensors of rank $3N$ constructed as linear combinations of products of Kronecker symbols δ_A^B and Minkowski metrics η_{AB} with coefficients which are functions of the scalar field ϕ and its kinetic term X. A disformal transformation then induces a transformation on the space of these coefficient functions, similarly to the transformation (6.1.22) derived for conformal transformations. These transformations are rather tedious to calculate in general. As an illustrative example, we worked out all possible terms up to $N = 2$ [72].

6.2.3 Generalization to multiple scalar fields

By extending our work [H9], also disformal transformations can be generalized to the case of multiple scalar fields $\phi^{\mathfrak{a}}$, which may be interpreted as the coordinates of a point on a field value manifold F, similarly to the case of conformal transformations. In this case, the transformation of the tetrad can be written as

$$\tilde{\theta}^A{}_\mu = \mathfrak{C}(\boldsymbol{\phi}, \boldsymbol{X})\theta^A{}_\mu + \mathfrak{D}_{\mathfrak{a}\mathfrak{b}}(\boldsymbol{\phi}, \boldsymbol{X}) g^{\nu\rho}\theta^A{}_\rho \partial_\mu\phi^{\mathfrak{a}}\partial_\nu\phi^{\mathfrak{b}}\,, \tag{6.2.15}$$

which is complemented by the transformation of the scalar field given by a diffeomorphism $\boldsymbol{f} : F \to F$ of the scalar field space. While the latter already has a straightforward geometric interpretation, the former needs clarification. Recall from section 2.2.5 that we interpret the scalar field multiplet $\boldsymbol{\phi}$ as a section of the trivial bundle $M \times F$, or equivalently as a map $\boldsymbol{\phi} : M \to F$. The differential $\boldsymbol{\phi}_* : TM \to TF$ of the latter is a vector bundle morphism covering $\boldsymbol{\phi}$, i.e., to every $v \in TM$ with $\tau_M(v) = x$ it assigns $\boldsymbol{\phi}_*(v) \in TF$ with $\tau_F(\boldsymbol{\phi}_*(v)) = \boldsymbol{\phi}(x)$. However, the fiber of TF over $\boldsymbol{\phi}(x)$ is the same as the fiber of the pullback bundle

$$\boldsymbol{\phi}^*TF = \{(x, u) \in M \times TF, \boldsymbol{\phi}(x) = \tau_F(x)\} \tag{6.2.16}$$

over x, where the projection is simply given by $(x, u) \mapsto x$. Hence, we may equivalently regard the differential $\boldsymbol{\phi}_*$ as a vector bundle morphism

$$\begin{array}{rcccl} (\tau_M, \boldsymbol{\phi}) & : & TM & \to & \boldsymbol{\phi}^*TF \\ & & v & \mapsto & (\tau_M(v), \boldsymbol{\phi}_*(v)) \end{array} \tag{6.2.17}$$

covering the identity on M. Further, a vector bundle morphism covering the identical is again equivalently described by a section of the tensor product bundle $T^*M \otimes \boldsymbol{\phi}^*TF$. Finally, realizing that the inverse of the metric g is a symmetric tensor of rank $(2, 0)$, and hence a section of $\mathrm{Sym}(TM \otimes TM)$, we can contract it with two copies of the aforementioned tensor bundle section, which gives us a section of $\mathrm{Sym}(\boldsymbol{\phi}^*TF \otimes \boldsymbol{\phi}^*TF)$. The field $\boldsymbol{X}$ we introduced in the definition (6.2.15) is simply this section, up to a factor $-\frac{1}{2}$, so that its coordinate expression

$$\boldsymbol{X} = X^{\mathfrak{a}\mathfrak{b}}\eth_{\mathfrak{a}} \otimes \eth_{\mathfrak{b}} = -\frac{1}{2}g^{\mu\nu}\phi^{\mathfrak{a}}_{,\mu}\phi^{\mathfrak{b}}_{,\nu}\eth_{\mathfrak{a}} \otimes \eth_{\mathfrak{b}}\,, \tag{6.2.18}$$

matches with the definition (6.1.26) of the kinetic term $X^{\mathfrak{a}\mathfrak{b}}$. Here $\eth_{\mathfrak{a}}$ denote the coordinate vector fields on F. While this coordinate expression is of course rather simple to display, it is still instructive to understand that it indeed has a fundamental geometric interpretation. With this interpretation in mind, it is now clear that the pair $(\boldsymbol{\phi}, \boldsymbol{X})$ simply constitutes a map from M to $\mathrm{Sym}(TF \otimes TF)$, which is constructed from the metric g and the scalar fields $\boldsymbol{\phi}$. The disformal transformations (6.2.15) are thus parametrized by functions

$$\mathfrak{C} : \mathrm{Sym}(TF \otimes TF) \to \mathbb{R}\,, \quad \mathfrak{D} : \mathrm{Sym}(TF \otimes TF) \to T^*F \otimes T^*F\,, \tag{6.2.19}$$

where the latter can be shown by a very similar construction, interpreting the values of $\mathfrak{D}_{\mathfrak{ab}}$ as components of symmetric tensors of rank $(0, 2)$ on F. Note that $\mathfrak{D}$ is in general only a bundle morphism over F, but not necessarily a vector bundle morphisms, even though its domain and codomain are vector bundles over F. These functions must further be restricted by the demand that the transformed tetrad (6.2.15) is non-degenerate, which leads to the condition

$$\det(\mathfrak{C}\delta^\mu_\nu + \mathfrak{D}_{\mathfrak{ab}} g^{\mu\rho}\partial_\nu\phi^{\mathfrak{a}}\partial_\rho\phi^{\mathfrak{b}}) > 0\,. \tag{6.2.20}$$

Evaluating this expression, we find that it becomes significantly more involved than in the case of a single scalar field, so that we omit the result here. Similarly, also the group structure, comprised of the product and inverse of disformal transformations, becomes rather involved in this most general case.

It is, nevertheless, instructive to take a closer look at infinitesimal disformal transformations, in order to see how these are related to the framework laid out in section 2.3.3. For this purpose one may consider a one-parameter group of disformal transformation with parameter $t \in \mathbb{R}$, which is defined by the parameter functions $(\mathfrak{C}_t, \mathfrak{D}_t, \boldsymbol{f}_t)$. It follows from the conditions on a one-parameter group that for $t = 0$ one finds the neutral element

$$(\mathfrak{C}_0, \mathfrak{D}_0, \boldsymbol{f}_0) = (1, 0, \mathrm{id}_F)\,, \tag{6.2.21}$$

where 1 denotes the constant function on $\mathrm{Sym}(TF \otimes TF)$ and 0 likewise denotes the constant bundle morphism mapping to the zero section. The generator of this one-parameter group consists of a real function $\mathfrak{C}'_0 : \mathrm{Sym}(TF \otimes TF) \to \mathbb{R}$, a bundle morphism $\mathfrak{D}'_0 : \mathrm{Sym}(TF \otimes TF) \to T^*F \otimes T^*F$ (where we identified the vertical tangent spaces with the fibers of the codomain) and a vector field $\boldsymbol{f}'_0 : F \to TF$. To study the action of these generators on the field space, we denote the coordinates on F by $(s^{\mathfrak{a}})$, so that we have coordinates $(x^\mu, f_A{}^\mu, s^{\mathfrak{a}})$ on the total space of the bundle $E = \mathrm{GL}(M) \times F$. For the corresponding jet bundle coordinates, we use the notation introduced in section 2.1.3. Note that here $f_A{}^\mu$ denote the frame components, while we defined the disformal transformation (6.2.15) using the coframe $\theta^A{}_\mu$. In coordinates, we thus have the transformation

$$\tilde{f}^{-1\,A}{}_\mu = \mathfrak{C}_t f^{-1\,A}{}_\mu + \mathfrak{D}_{t\,\mathfrak{ab}}\eta^{AB} f_B{}^\nu s^{\mathfrak{a}}{}_\mu s^{\mathfrak{b}}{}_\nu\,, \tag{6.2.22}$$

where $\mathfrak{C}_t$ and $\mathfrak{D}_t$ depend on the coordinate expressions $s^{\mathfrak{a}}$ and

$$-\frac{1}{2}\eta^{AB} f_A{}^\mu f_B{}^\nu s^{\mathfrak{a}}{}_\mu s^{\mathfrak{b}}{}_\nu \tag{6.2.23}$$

defined on the first jet bundle. Differentiating with respect to the parameter t, and using

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{f}^{-1\,A}{}_\mu\bigg|_{t=0} = -\,\tilde{f}^{-1\,A}{}_\nu \tilde{f}^{-1\,B}{}_\mu \frac{\mathrm{d}}{\mathrm{d}t}\tilde{f}_B{}^\nu\bigg|_{t=0}\,, \tag{6.2.24}$$

we find that the action of the one-parameter group of disformal transformations is generated by the evolutionary vector field

$$\begin{aligned} \mathbf{X} &= \boldsymbol{f}'^{\mathfrak{a}}_0 \eth_{\mathfrak{a}} - f_A{}^\nu f_B{}^\mu \left(\mathfrak{C}'_0 f^{-1\,B}{}_\nu + \mathfrak{D}'_{0\,\mathfrak{ab}}\eta^{BC} f_C{}^\rho s^{\mathfrak{a}}{}_\nu s^{\mathfrak{b}}{}_\rho\right)\bar{\partial}^A{}_\mu \\ &= \boldsymbol{f}'^{\mathfrak{a}}_0 \eth_{\mathfrak{a}} - \left(\mathfrak{C}'_0 f_A{}^\mu + \mathfrak{D}'_{0\,\mathfrak{ab}} f^A{}_\nu g^{\mu\rho} s^{\mathfrak{a}}{}_\nu s^{\mathfrak{b}}{}_\rho\right)\bar{\partial}^A{}_\mu\,. \end{aligned} \tag{6.2.25}$$

With this expression at hand, one may study the transformation of any Lagrangian, such as the ones constructed in section 6.2.2, under infinitesimal disformal transformations, using the framework displayed in section 2.3.4. Depending on the choice of the Lagrangian, this may become rather involved, and we will not further pursue this calculation here.

7 Summary and outlook

In this thesis we have discussed the relevance of symmetry transformations in the geometric formulation of gravity theories. In particular, we have:

1. extended the notion of invariance under the action of a symmetry group from pseudo-Riemannian geometry to more general geometries used in gravity theory;
2. determined the most general classes of invariant geometries for different symmetry groups;
3. studied the transformation of perturbed geometries under the action of infinitesimal diffeomorphisms on the background geometry and derived gauge-invariant quantities;
4. discussed the transformation of geometric fields under group actions on the field space and the introduced transformation of the action functional of gravity theories.

Despite having very diverse applications in gravity theory, we have cast these different topics into a common and concise differential geometric framework, since they are based on the same underlying mathematical notions. Having such a framework at hand does not only provide better insight and understanding of the role of symmetry in the theory of gravity, but can also serve as a guiding principle for future studies in this field.

Besides the particular geometries we studied in this thesis, also other, potentially more general notions of geometry have found application in gravitational field theories. These include higher order gauge theory [4], symmetric teleparallel gravity [133, 109], general teleparallel gravity [10, 22] as well as Hamilton and Lagrange geometries [132, 2]. Even more general geometric structures are employed in quantum gravity. Understanding the action of transformation groups on these structures, finding symmetric solution, revealing symmetries of gravity theories and deriving observational consequences is therefore a crucial task.

The notions of symmetry derived for the aforementioned geometries and for the geometries discussed in this thesis may further be used to derive the most general gravitational field configurations which are invariant under the action of various other transformation groups. The most notable contender classes of symmetry groups would be more general types of cosmological symmetry, giving rise to the homogeneous, but not necessarily isotropic Bianchi cosmologies [13], as well as planar symmetry, which is relevant for the study of exact plane wave solutions. Once such exact symmetric solutions are found, one may furher proceed and study perturbations of these geometries, as well as their transformation under the symmetry group of the background spacetime, and gauge-invariant perturbation variables may be constructed similar to those in Riemannian and teleparallel geometries.

Finally, the general framework discussed in this thesis, which covers also symmetries of gravity theories under transformations of their field space, allows for further applications. Besides the disformal transformations of teleparallel geometry generated by scalar fields, also vector fields may be included [115], and more general transformations of the connection in the framework of metric-affine geometry may be considered [106, 105].

A Included publications

[H1] [65] Manuel Hohmann, "Spacetime and observer space symmetries in the language of Cartan geometry", J. Math. Phys. **57** (2021) no. 8, 082502 [arXiv:1505.07809].

[H2] [85] Manuel Hohmann, Laur Järv, Martin Krššák, and Christian Pfeifer, "Modified teleparallel theories of gravity in symmetric spacetimes", Phys. Rev. D **100** (2019) no. 8, 084002 [arXiv:1901.05472].

[H3] [76] Manuel Hohmann, "Metric-affine Geometries With Spherical Symmetry", Symmetry **12** (2020) no. 3, 453 [arXiv:1912.12906].

[H4] [77] Manuel Hohmann, "Complete classification of cosmological teleparallel geometries", Int. J. Geom. Meth. Mod. Phys. **18** (2021) no. supp01, 2140005 [arXiv:2008.12186].

[H5] [78] Manuel Hohmann, "General cosmological perturbations in teleparallel gravity", Eur. Phys. J. Plus **136** (2021) no. 1, 65 [arXiv:2011.02491].

[H6] [74] Manuel Hohmann, "Gauge-invariant approach to the parametrized post-Newtonian formalism", Phys. Rev. D **101** (2020) no. 2, 024061 [arXiv:1910.09245].

[H7] [73] Manuel Hohmann, "Spherical harmonic d-tensors", Int. J. Geom. Meth. Mod. Phys. **16** (2019) no. supp02, 1941002 [arXiv:1812.11169].

[H8] [71] Manuel Hohmann, "Scalar-torsion theories of gravity III: analogue of scalar-tensor gravity and conformal invariants", Phys. Rev. D **98** (2018) no. 6, 064004 [arXiv:1801.06531].

[H9] [72] Manuel Hohmann, "Disformal Transformations in Scalar-Torsion Gravity", Universe **5** (2019) no. 7, 167 [arXiv:1905.00451]

These publications have been chosen since they are either single-authored or the contribution of the thesis author constituted a substantial part, and since they form a consistent line of investigation focused on a common research topic which is central for the thesis author's work.

All included publications are single-authored publications by the thesis author, except for [H2]. The contribution of the thesis author to the latter was to derive the symmetry condition from a more general condition in the previous work [H1], to construct a method for solving this condition in the Weitzenböck gauge, to construct the Lie algebra homomorphisms which are the starting point of this solution method, and to prove the non-existence of such a homomorphism for the case of non-flat maximal symmetry.

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I Publication H1

Manuel Hohmann

Spacetime and observer space symmetries in the language of Cartan geometry

J. Math. Phys. **57** (2021) no. 8, 082502 [arXiv:1505.07809].

II Publication H2

Manuel Hohmann, Laur Järv, Martin Kršśák, and Christian Pfeifer

Modified teleparallel theories of gravity in symmetric spacetimes

Phys. Rev. D **100** (2019) no. 8, 084002 [arXiv:1901.05472].

III Publication H3

Manuel Hohmann

Metric-affine Geometries With Spherical Symmetry

Symmetry **12** (2020) no. 3, 453 [arXiv:1912.12906].

IV Publication H4

Manuel Hohmann

Complete classification of cosmological teleparallel geometries

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V Publication H5

Manuel Hohmann

General cosmological perturbations in teleparallel gravity

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VI Publication H6

Manuel Hohmann

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VII Publication H7

Manuel Hohmann

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VIII Publication H8

Manuel Hohmann

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Phys. Rev. D **98** (2018) no. 6, 064004 [arXiv:1801.06531].

IX Publication H9

Manuel Hohmann

Disformal Transformations in Scalar-Torsion Gravity

Universe **5** (2019) no. 7, 167 [arXiv:1905.00451]