

Local symmetries and physical degrees of
freedom in $f(T)$ gravity:
a Dirac Hamiltonian constraint analysis

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based on work with Milutin Blagojević

Phys Rev D **102** (2020) 064025, arXiv:2006.15303

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Abstract: Blagojević & Nester, PRD **102** (2020) 064025, arXiv:2006.15303

- ▶ For $f(\mathbb{T})$ gravity, the status of local Lorentz invariance and the number of physical degrees of freedom have been controversial issues.
- ▶ We did a detailed Dirac constraint Hamiltonian analysis;
- ▶ there are several scenarios describing how local Lorentz invariance can be broken.
- ▶ **Generically there are 5 physical degrees of freedom;**
- ▶ in D dimensions, there are $D(D - 3)/2 + (D - 1)$.
- ▶ **As expected, the theory is vulnerable to having problematical propagating modes.**
- ▶ We compared our results with those in the literature.
- ▶ Diffeomorphism invariance was explicitly confirmed.

The nature of this presentation

- ▶ This material is all excerpted from our paper,
M. Blagojević and J. M. Nester,
“Local symmetries and physical degrees of freedom in $f(T)$ gravity: a Dirac Hamiltonian constraint analysis,”
Phys. Rev. D **102** (2020) 064025 [arXiv:2006.15303 [gr-qc]].
- ▶ I want to introduce to you this rather long technical work,
- ▶ Here I tried to include the parts that might be of more interest to you.
- ▶ Excuse me, for some of the material included in this file, especially the technical appendices at the end, I did not get around to refining the editing of eqn numbers, etc. My apologies.

Background and notation

- ▶ Our conventions are as follows.
- ▶ latin indices (i, j, \dots) are the local Lorentz indices, greek indices (μ, ν, \dots) are the coordinate indices, and both run over $0, 1, 2, 3$;
- ▶ the orthonormal frame (tetrad) is $\vartheta^i = \vartheta^i{}_{\mu} dx^{\mu}$ (1-form), $\vartheta = \det(\vartheta^i{}_{\mu})$, the dual basis (frame) is $e_i = e_i{}^{\mu} \partial_{\mu}$,
- ▶ the metric components in the local Lorentz and coordinate basis are $g_{ij} = (1, -1, -1, -1)$ and $g_{\mu\nu} = g_{ij} \vartheta^i{}_{\mu} \vartheta^j{}_{\nu}$, respectively,
- ▶ and ε_{ijmn} is the totally antisymmetric symbol with $\varepsilon_{0123} = 1$.
- ▶ The earlier speakers have already introduced torsion, and the teleparallel equivalent to GR (following Hehl we use the name GR_{\parallel}), and the specific teleparallel theory of interest here: $f(\mathbb{T})$.
- ▶ The torsion 2-form is $T^i := d\vartheta^i + \omega^i{}_j \vartheta^j$, and we shall use *teleparallel frames*, so the spin coefficients vanish.

From teleparallel to $f(\mathbb{T})$ gravity

- ▶ In our analysis of the teleparallel theories, we relied, as did the earlier investigators [Li Miao Miao (2011) and Ferraro & Guzmán (2018)], on the pure tetrad formalism, with vanishing spin connection. The general (parity even) TG Lagrangian is defined in terms of 3 independent quadratic invariants,

$$L_{TG} = \vartheta \mathcal{L}_{TG}, \quad \mathcal{L}_{TG} := a_0 T^{ijk} (h_1 T_{ijk} + h_2 T_{jik} + h_3 g_{ij} T^m{}_{mk}),$$

where $a_0 = 1/16\pi G$.

- ▶ For the special choice of parameters $(h_1, h_2, h_3) = (1/4, 1/2, -1)$, one obtains the GR_{\parallel} Lagrangian

$$L_T = \vartheta \mathcal{L}_T, \quad \mathcal{L}_T = \frac{1}{4} a_0 T^{ijk} (T_{ijk} + 2T_{jik} - 4g_{ij} V_k), \quad V_k := T^m{}_{mk},$$

- ▶ The covariant momentum,

$$H_{ijk} := \frac{\partial L_T}{\partial T^{ijk}} = \vartheta \mathcal{H}_{ijk}, \quad \mathcal{H}_{ijk} = a_0 (T_{ijk} + 2T_{[kj]i} - 4g_{i[j} V_{k]}) ,$$

plays an important (technical) role in the canonical analysis of GR_{\parallel} . Variation of L_T wrt $\vartheta^i{}_{\nu}$ yields the gravitational field equations, equivalent to the GR field equations in vacuum.

- ▶ In recent decades (mainly motivated by cosmological puzzles), alternative gravitational models, based on modifications of GR or its non-Riemannian extensions, have been proposed. In particular, $f(R)$ gravity motivates one to introduce an analogous $f(\mathbb{T})$ Lagrangian,

$$\mathcal{L}_{fT} := f(\mathbb{T}), \quad \mathbb{T} := \mathcal{L}_T \equiv \frac{1}{4} a_0 T^{ijk} (T_{ijk} + 2T_{jik} - 4g_{ij} V_k).$$

\mathbb{T} is the teleparallel counterpart of the Riemannian scalar R .

The Hamiltonian approach to teleparallel gravity

- ▶ Twenty years ago Hamiltonian analysis of covariant teleparallel gravity [Blagojević & Nikolić, PRD **62** (2000) 024021, arXiv:0002022;] [Blagojević & Vasilic, CQG **17** (2000) 3785, arXiv:0006080] showed that TG always has gauge generators associated with the local Poincaré symmetry of the underlying Riemann-Cartan spacetime: the local translations and the Lorentz symmetry of the frame-connection pair $(\vartheta^i, \omega^{ij})$.
- ▶ But for the one special case of GR_{\parallel} , there is an additional local Lorentz symmetry that acts on the frame alone. This *pure frame local Lorentz symmetry* is absent for any other choice of the teleparallel Lagrangian.
- ▶ After fixing the gauge $\omega^{ij} = 0$, the local Lorentz symmetry of the frame-connection pair is broken. The gauge symmetries of the generic tetrad form of TG are only local translations, whereas in the special case of GR_{\parallel} , the pure frame Lorentz symmetry survives as a valid gauge symmetry of the theory.
- ▶ $f(\mathbb{T})$ gravity is more complicated. One might expect breaking of the pure frame local Lorentz symmetry, but it is not a priori clear whether the violation is complete or only partial.

- ▶ To simplify the Hamiltonian analysis of $f(\mathbb{T})$ gravity, we [as did Li, Miao, Miao (2011), Ferraro & Guzmán (2018)] find it convenient to represent $f(\mathbb{T})$ as the Legendre transform of a function $V(\phi)$

$$L = \vartheta \mathcal{L}, \quad \mathcal{L} := \phi \mathbb{T} - V(\phi),$$

where ϕ is an auxiliary scalar field.

- ▶ The new Lagrangian $\mathcal{L} = \mathcal{L}(\phi, \mathbb{T})$ is dynamically equivalent to $f(\mathbb{T})$. Using the relation $\mathbb{T} = V'(\phi)$ obtained from $\delta \mathcal{L} / \delta \phi = 0$, one can express ϕ as a function of \mathbb{T} , $\phi = \phi(\mathbb{T})$, provided V' is invertible. $V'(\phi)$ is invertible if $V''(\phi) \neq 0$.

Our analysis is based on this Lagrangian, with the dynamical variables $(\vartheta^i{}_\mu, \phi)$. The covariant momentum takes the convenient form

$$H_{ijk}^f := \frac{\partial L^f}{\partial T^{ijk}} = \phi H_{ijk}, \quad \mathcal{H}_{ijk}^f := \phi \mathcal{H}_{ijk},$$

which is obtained from the GR_{\parallel} expression by the simple rule $a_0 \rightarrow a_0 \phi$. Variation wrt $\vartheta^i{}_\mu$ and ϕ yields the vacuum field equations,

$$\begin{aligned} \mathcal{E}_i{}^\nu &:= -\frac{\delta L^f}{\delta \vartheta^i{}_\nu} = \nabla_\mu (\phi H_i{}^{\mu\nu}) + T_{mni} (\phi H^{mn\nu}) - e_i{}^\nu L^f \\ &\equiv (\partial_\mu \phi) H_i{}^{\mu\nu} + \phi \left[\nabla_\mu H_i{}^{\mu\nu} + T_{mni} H^{mn\nu} - e_i{}^\nu \vartheta \mathbb{T} \right] + e_i{}^\nu \vartheta V(\phi) = 0, \\ \mathcal{E}_\phi &:= \vartheta \left[\mathbb{T} - \partial_\phi V(\phi) \right] = 0. \end{aligned}$$

The first equation can be transformed into

$$\mathcal{E}^{ik} = (\partial_\mu \phi) H^{ik\mu} - \phi \left[2a_0 \vartheta \left(R^{ik}(\tilde{\omega}) - \frac{1}{2} g^{ik} R(\tilde{\omega}) \right) \right] + g^{ik} \vartheta V(\phi) = 0,$$

where $\tilde{\omega}$ is the Riemannian connection. Its antisymmetric part is

$$\boxed{\mathcal{E}^{[ik]} = H^{[ik]\mu} \partial_\mu \phi = 0}.$$

In GR_{\parallel} , these six equations are trivial.

Hamiltonian form of $f(\mathbb{T})$ gravity

- ▶ The Dirac Hamiltonian formalism for a system with constraints is a good approach for analyzing both local symmetries and dynamical d.o.f. in any theory including $f(\mathbb{T})$ gravity.
- ▶ We rely on the $(1 + 3)$ decomposition of spacetime, whose basic aspects can be characterized by two simple properties:
 - (p1) at each point of a spatial hypersurface $\Sigma : x^0 = \text{const.}$, one can define a unit timelike vector $\mathbf{n} = (n_k)$, orthogonal to Σ ;
 - (p2) any spacetime vector $\mathbf{V} = (V_k)$ can be decomposed into a component $V_\perp := n^k V_k$ along \mathbf{n} and another component $V_{\bar{k}} := V_k - n_k V_\perp$ laying in the tangent space of (“parallel” to) Σ . As a consequence, $n^k V_{\bar{k}} = 0$.
- ▶ Also we use $\varepsilon_{\bar{i}\bar{j}\bar{k}}$ as the Euclidean epsilon symbol with $\varepsilon_{\bar{1}\bar{2}\bar{3}} = 1$, and δ_{mnl}^{ijk} is the generalized Kronecker symbol.

Primary constraints

The canonical momenta (π_i^μ, π_ϕ) associated to the basic Lagrangian variables (ϑ^i_μ, ϕ) are

$$\pi_i^\mu = \frac{\partial L}{\partial T^i_{0\mu}} = \phi H_i^{0\mu}, \quad \pi_\phi = \frac{\partial \mathcal{L}}{\partial_0 \phi} = 0.$$

There are $4 + 1$ obvious primary constraints $\pi_i^0 \approx 0$, $\pi_\phi \approx 0$, independent of the particular values of the coupling constants. The π_i^α can be expressed in terms of the parallel canonical momentum:

$$\begin{aligned} \hat{\pi}_{i\bar{k}} &:= \pi_i^\alpha \vartheta_{k\alpha} = \phi J \mathcal{H}_{i\perp\bar{k}}, \\ \mathcal{H}_{i\perp\bar{k}} &= a_0 \left[T_{i\perp\bar{k}} + (T_{\bar{k}\perp i} - T_{\perp\bar{k}i}) - 2(n_i V_{\bar{k}} - g_{i\bar{k}} V_\perp) \right], \end{aligned}$$

where $J := \vartheta/N$. To clarify the constraint content we introduce $P_{i\bar{k}}$,

$$P_{i\bar{k}} := \hat{\pi}_{i\bar{k}}/J - \phi \mathcal{H}_{i\perp\bar{k}}(0) = \phi \mathcal{H}_{i\perp\bar{k}}(1),$$

where $\mathcal{H}_{i\perp\bar{k}}(0)$ does not depend on the “velocities” $T_{i\perp\bar{k}}$, and $\mathcal{H}_{i\perp\bar{k}}(1)$ is linear in them. The 3D rotational irreducible decomposition of $P_{i\bar{k}}$ yields

$$\begin{aligned}
P_{\perp\bar{k}} &\equiv \hat{\pi}_{\perp\bar{k}}/J + 2a_0\phi T^{\bar{m}}_{\bar{m}\bar{k}} \approx 0, \\
{}^A P_{\bar{i}\bar{k}} &\equiv {}^A \hat{\pi}_{\bar{i}\bar{k}}/J - a_0\phi T_{\perp\bar{i}\bar{k}} \approx 0, \quad \text{antisymmetric} \\
P^{\bar{m}}_{\bar{m}} &\equiv \hat{\pi}^{\bar{m}}_{\bar{m}}/J = 4a_0\phi T^{\bar{m}}_{\bar{m}\perp}, \\
{}^T P_{\bar{i}\bar{k}} &\equiv {}^T \hat{\pi}_{\bar{i}\bar{k}}/J = 2a_0\phi {}^T T_{\bar{i}\perp\bar{k}}, \quad \text{symmetric-traceless}
\end{aligned}$$

The first two obviously define 6 additional primary constraints. To simplify calculations we represent these primary constraints in a compact form:

$$\boxed{C_{ik} = \mathcal{H}_{ik} + a_0\phi B_{ik}},$$

$$\begin{aligned}
\mathcal{H}_{ik} &:= \hat{\pi}_{i\bar{k}} - \hat{\pi}_{k\bar{i}} = 2\pi_{[i}^{\alpha} \vartheta_{k]\alpha}, \\
B_{ik} &:= \partial_{\alpha} B_{ik}^{0\alpha}, \quad B_{ik}^{0\alpha} \equiv \varepsilon_{ikmn} \vartheta^m_{\beta} \vartheta^n_{\gamma},
\end{aligned}$$

This follows from the identity $B_{ik} = -2J(T^{\perp}_{\bar{i}\bar{k}} - n_i T^{\bar{m}}_{\bar{m}\bar{k}} + n_k T^{\bar{m}}_{\bar{m}\bar{i}})$.

- ▶ The existence of the constraints C_{ij} is caused by the special values of the coupling constants in the GR_{\parallel} Lagrangian.
- ▶ The remaining two relations are not constraints, they relate the velocities $T_{i\perp\bar{k}}$ to the canonical momenta $\hat{\pi}_{i\bar{k}}$.

Hamiltonians

One can rewrite the canonical Hamiltonian,

$$H_c := \pi_i^\alpha \partial_0 \vartheta^i_\alpha - b\mathcal{L},$$

in the standard Dirac-ADM form:

$$H_c = N\mathcal{H}_\perp + N^\beta \mathcal{H}_\beta + \partial_\alpha D^\alpha,$$

$$\mathcal{H}_\perp := \hat{\pi}_i^{\bar{m}} T^i_{\perp\bar{m}} - J\mathcal{L} - n^i \partial_\alpha \pi_i^\alpha,$$

$$\mathcal{H}_\beta := \pi_i^\alpha T^i_{\beta\alpha} - \vartheta^i_\beta \partial_\alpha \pi_i^\alpha,$$

$$D^\alpha := \pi_i^\alpha \vartheta^i_0.$$

Here, the lapse and shift functions $N = n_k \vartheta^k_0$ and $N^\alpha = e_{\bar{k}}^\alpha \vartheta^k_0$, respectively, are linear in the unphysical variables ϑ^k_0 .

The lapse Hamiltonian \mathcal{H}_\perp is the only dynamical part as it depends on the Lagrangian.

To eliminate the velocities $T_{i\perp\bar{k}}$ from \mathcal{H}_\perp , we use the relation

$$\hat{\pi}_i^{\bar{m}} T^i_{\perp\bar{m}} - J\mathcal{L} = \frac{1}{2} J P^{i\bar{m}} T_{i\perp\bar{m}} - J\bar{\mathcal{L}},$$

where $\bar{\mathcal{L}} := \mathcal{L}(0) \equiv \phi\bar{\mathbb{T}} - V(\phi)$. Then, inserting the irreducible decomposition

$$P^{i\bar{m}} T_{i\perp\bar{m}} = \frac{1}{3} P^{\bar{m}}_{\bar{m}} T^{\bar{n}}_{\perp\bar{n}} + {}^T P_{\bar{i}\bar{k}} T^{\bar{i}\perp\bar{k}} + \left[{}^A P_{\bar{i}\bar{k}} {}^A T^{\bar{i}\perp\bar{k}} + P_{\perp\bar{k}} T^{\perp\perp\bar{k}} \right],$$

and eliminating the velocities gives

$$\begin{aligned}\mathcal{H}_\perp &\approx \frac{1}{2a_0\phi} P^2 - J(\phi\bar{\mathbb{T}} - V) - n^i \partial_\alpha \pi_i^\alpha =: \check{\mathcal{H}}_\perp, \\ P^2 &:= \frac{1}{2J} \left[\hat{\pi}^{(\bar{m}\bar{n})} \hat{\pi}^{(\bar{m}\bar{n})} - \frac{1}{2} (\hat{\pi}^{\bar{m}}_{\bar{m}})^2 \right], \\ \bar{\mathbb{T}} &:= \frac{1}{4} a_0 \left(T_{i\bar{m}\bar{n}} T^{i\bar{m}\bar{n}} + 2T_{\bar{i}\bar{m}\bar{n}} T^{\bar{m}\bar{i}\bar{n}} - 4T^{\bar{m}}_{\bar{m}\bar{k}} T_{\bar{n}}^{\bar{n}\bar{k}} \right).\end{aligned}$$

Apart from the V term, the rest of $\check{\mathcal{H}}_\perp$ is obtained from the GR_\parallel expression by $a_0 \rightarrow a_0\phi$. The Hamiltonian dynamics is described by the total Hamiltonian

$$H_T = \check{H}_c + u^i{}_0 \pi_i^0 + u_\phi \pi_\phi + \frac{1}{2} u^{ij} C_{ij},$$

where u' s are, at this stage, arbitrary Hamiltonian multipliers. Their dynamical interpretation is as usual from the Hamiltonian equations of motion: the “missing velocities”

$$u_\phi = \partial_0 \phi = N \partial_\perp \phi + N^\alpha \partial_\alpha \phi, \quad u^{\perp\bar{n}} = N T^{\perp\perp\bar{n}}, \quad u^{\bar{m}\bar{n}} = N^A T^{\bar{m}\perp\bar{n}}.$$

The complete dynamical Hamiltonian \mathcal{H}_\perp , the Legendre transform of \mathcal{L} with respect to the velocity $T^i{}_{\perp\bar{m}}$, is

$$\mathcal{H}_\perp = \check{\mathcal{H}}_\perp + \frac{1}{2} \hat{u}^{mn} C_{mn}, \quad \hat{u}^{mn} := N^{-1} u^{mn}.$$

Simultaneously, the expression for the total Hamiltonian is simplified,

$$\boxed{H_T = H_c + u^i{}_0 \pi_i^0 + u_\phi \pi_\phi}.$$

Preservation of primary constraints

For consistency of the Hamiltonian analysis, every constraint φ appearing in the theory has to be preserved during dynamical evolution of the system, determined by the total Hamiltonian as

$$\chi := \partial_0 \varphi = \{\varphi, H_T\} \approx 0.$$

In the expressions for H_T , an integration over d^3x is implicitly understood. Let us now apply this condition to the primary constraints $\varphi_A = (\pi_i^0, \pi_\phi, C_{ij})$. The preservation condition for π_i^0 gives

$$\chi_i := -\{\pi_i^0, H_T\} = n_i \mathcal{H}_\perp + e_i^\alpha \mathcal{H}_\alpha \approx 0,$$

$$\chi_\perp = \mathcal{H}_\perp \approx 0, \quad \chi_\alpha = \mathcal{H}_\alpha \approx 0$$

For $\chi_\phi := \partial_0 \pi_\phi$,

$$\{\pi_\phi, \check{\mathcal{H}}_\perp\} = -\partial_\phi \check{\mathcal{H}}_\perp = \frac{1}{2a_0 \phi^2} P^2 + J(\bar{\mathbb{T}} - \partial_\phi V) =: F_\phi,$$

$$\{\pi_\phi, C_{ij}\} = -\partial_\phi C_{ij} = -F_{ij}, \quad F_{ij} := a_0 B_{ij},$$

$$\implies \boxed{\chi_\phi = N F_\phi - \frac{1}{2} u^{mn} F_{mn} \approx 0}.$$

We verified that the Lagrangian counterpart of χ_ϕ is $\vartheta(\mathbb{T} - \partial_\phi V)$.

The key preservation condition at the level of primary constraints is that of the Lorentz constraint C_{ij} , which is of particular importance for a proper understanding of the status of Lorentz invariance in $f(\mathbb{T})$ gravity. A direct calculation shows that the expression $\chi_{ij} := \{C_{ij}, H_T\}$ has the form

$$\chi_{ij} = G_{ij}{}^k (\partial_k \phi) \delta \approx 0,$$

$$G_{ij}{}^\perp := -2a_0 \vartheta (T^\perp{}_{\bar{i}\bar{j}} - n_i \bar{V}_{\bar{j}} + n_j \bar{V}_{\bar{i}}) = NF_{ij},$$

$$G_{ij}{}^{\bar{k}} := 2a_0 J \delta_{ijm}^\perp{}^{\bar{k}\bar{n}} u^m{}_{\bar{n}} - N \left[\phi^{-1} g^{\bar{k}\bar{m}} (n_i \hat{\pi}_{(\bar{m}\bar{j})} - n_j \hat{\pi}_{(\bar{m}\bar{i})}) + a_0 J \delta_{\bar{i}\bar{j}\bar{r}}^{\bar{k}\bar{m}\bar{n}} T^{\bar{r}}{}_{\bar{m}\bar{n}} \right].$$

- ▶ The Hamiltonian transcription of the six Lagrangian equations

$$\mathcal{E}^{[ik]} = H^{[ik]\mu} \partial_\mu \phi = 0 \text{ is } \chi_{ij}.$$

- ▶ From the analysis presented in Appendix C, one can explicitly determine the Poisson bracket (PB) of C_{ij} with itself. The result can be written in the form

$$u^{mn} \{C_{ij}, C_{mn}\} = 2u^{mn} (g_{jm} C_{in} - g_{im} C_{jn}) + 4a_0 J \phi_{\bar{k}} \delta_{ijm}^\perp{}^{\bar{k}\bar{n}} u^m{}_{\bar{n}}.$$

- ▶ The presence of the nontrivial last term shows that at least some components of C_{ij} are second class constraints, which means that local Lorentz invariance is at least partially broken.

- ▶ The preservation of the primary constraints φ_A leads to the corresponding secondary constraints,

$$\chi_A := (\check{\mathcal{H}}_{\perp}, \mathcal{H}_{\alpha}; \chi_{\phi}, \chi_{ij}).$$

- ▶ Since $f(\mathbb{T})$ gravity is invariant under local translations (diffeomorphisms), the Hamiltonians $(\check{\mathcal{H}}_{\perp}, \mathcal{H}_{\alpha})$ (or their suitable deformations) are expected to be first class (FC).
- ▶ On the other hand, the constraints (χ_{ϕ}, χ_{ij}) define seven conditions on the seven multipliers (u_{ϕ}, u^{mn}) . These conditions will play a prominent role in the forthcoming analysis of local Lorentz invariance, as well as in counting the physical d.o.f.
- ▶ The results found in the present analysis based on $\phi\mathbb{T} - V(\phi)$ is compared with the canonical structure based on the standard Lagrangian $\mathcal{L}_{fT} = f(\mathbb{T})$ in Appendix D.

PB algebra between Hamiltonians

- ▶ We examine the PB algebra between the Hamiltonians $(\mathcal{H}_\perp, \mathcal{H}_\alpha)$, to understand their preservation conditions, and the status of diffeomorphisms invariance in $f(\mathbb{T})$ gravity.
- ▶ In the generic TG, the Hamiltonians satisfy the same PB algebra as in GR,

$$\begin{aligned}\{\mathcal{H}_\alpha, \mathcal{H}'_\beta\} &= (\mathcal{H}_\beta \partial_\alpha - \mathcal{H}'_\alpha \partial'_\beta) \delta, \\ \{\mathcal{H}_\alpha, \mathcal{H}'_\perp\} &= \mathcal{H}_\perp \partial_\alpha \delta, \\ \{\mathcal{H}_\perp, \mathcal{H}'_\perp\} &\approx -({}^3g^{\alpha\beta} \mathcal{H}_\beta + {}^3g'^{\alpha\beta} \mathcal{H}'_\beta) \partial_\alpha \delta.\end{aligned}$$

An elegant proof can be found in Mitrić [arXiv:1910.02810].

- ▶ The case of $f(\mathbb{T})$ gravity requires further generalizations. \mathcal{H}_α and \mathcal{H}_\perp are effectively modified as

$$\begin{aligned}\bar{\mathcal{H}}_\alpha &:= \mathcal{H}_\alpha + \pi_\phi \partial_\alpha \phi, \\ \bar{\mathcal{H}}_\perp &:= \mathcal{H}_\perp + \pi_\phi \partial_\perp \phi.\end{aligned}$$

Here, \mathcal{H}_\perp is understood as the Legendre transform of \mathcal{L} ,

$$\mathcal{H}_\perp = \hat{\pi}_i^{\bar{m}} T^i_{\perp \bar{m}} - J\mathcal{L} - n^i \partial_\alpha \pi_i^\alpha.$$

With these modifications, the total Hamiltonian can be written in a more compact form as

$$\begin{aligned}H_T &= H_c + u^i{}_0 \pi_i^0 + \partial_\alpha D^\alpha, \\ H_c &= \vartheta^i{}_0 \bar{\mathcal{H}}_i = N \bar{\mathcal{H}}_\perp + N^\alpha \bar{\mathcal{H}}_\alpha,\end{aligned}$$

where the primary constraints C_{ij} and π_ϕ are hidden inside H_c .

As shown in Appendix E, the final result takes the standard form

$$\begin{aligned}\{\bar{\mathcal{H}}_\alpha, \bar{\mathcal{H}}'_\beta\} &\approx (\bar{\mathcal{H}}_\beta \partial_\alpha - \bar{\mathcal{H}}'_\alpha \partial'_\beta) \delta, \\ \{\bar{\mathcal{H}}_\alpha, \bar{\mathcal{H}}'_\perp\} &\approx \bar{\mathcal{H}}'_\perp \partial_\alpha \delta, \\ \{\bar{\mathcal{H}}_\perp, \bar{\mathcal{H}}'_\perp\} &\approx -({}^3g^{\alpha\beta} \bar{\mathcal{H}}_\beta + {}^3g'^{\alpha\beta} \bar{\mathcal{H}}'_\beta) \partial_\alpha \delta.\end{aligned}$$

Diffeomorphism invariance

- ▶ We now construct the Hamiltonian gauge generator for local translations in $f(\mathbb{T})$ gravity, based on Castellani's algorithm [L. Castellani, Symmetries of constrained Hamiltonian systems, Ann. Phys. (N.Y.) **143** (1982) 357–371], and use it to show that $(\bar{\mathcal{H}}_\perp, \bar{\mathcal{H}}_\alpha)$ are FC.
- ▶ If the local symmetries of a gauge theory are described only in terms of the gauge parameters $\xi^i(x)$ and their first derivatives, the canonical gauge generator has the form (integration over d^3x understood)

$$G = \dot{\xi}^i G_i^0 + \xi^i G_i^1,$$

where the phase-space functions G_i^0 and G_i^1 are determined by the conditions

$$\begin{aligned} G_i^0 &= C_{\text{PFC}}, \\ G_i^1 + \{G_i^0, H_T\} &= C_{\text{PFC}}, \\ \{G_i^1, H_T\} &= C_{\text{PFC}}. \end{aligned}$$

and C_{PFC} denotes a primary FC constraint. The construction starts with any primary FC constraint G_i^0 , and the algorithm describes how the corresponding G_i^1 should be determined.

Gauge generator of the generic TG

The total and canonical Hamiltonians of TG are given by

$$\begin{aligned}H_T &= H_c + u^i{}_0 \pi_i^0 + \partial_\alpha D^\alpha, \\H_c &:= N \mathcal{H}_\perp + N^\alpha \mathcal{H}_\alpha = \vartheta^i{}_0 \mathcal{H}_i,\end{aligned}$$

$$\text{where} \quad \mathcal{H}_i := n_i \mathcal{H}_\perp + e_i{}^\alpha \mathcal{H}_\alpha.$$

The PB algebra of the Hamiltonian constraints, obtained by the Legendre transform technique can be transformed to an equivalent form as

$$\{\mathcal{H}_i, \mathcal{H}_j\} = T^n{}_{ij} \mathcal{H}_n \delta.$$

Since the only primary FC constraints in TG are $\varphi_i = \pi_i^0$, we start by taking

$$G_i^0 = -\pi_i^0.$$

The condition (0.1a), combined with $\{G_i^0, H_T\} = \mathcal{H}_i$, implies

$$G_i^1 = -\mathcal{H}_i + \alpha_i^m \pi_m^0,$$

where the unknown coefficients α_i^m are determined by (0.1a),

$$\{G_i^1, H_T\} = -\vartheta^k{}_0 T^n{}_{ik} \mathcal{H}_n \delta - \alpha_i^n \mathcal{H}_n \delta = C_{\text{PFC}}.$$

Solving this condition for α_i^n yields

$$\alpha_i^n = -\vartheta^k{}_0 T^n{}_{ik}, \quad G_i^1 = -\mathcal{H}_i - \vartheta^k{}_0 T^n{}_{ik} \pi_n^0,$$

and the final gauge generator takes the form

$$G = -\dot{\xi}^i \pi_i^0 - \xi^i (\mathcal{H}_i + \vartheta^k{}_0 T^n{}_{ik} \pi_n^0).$$

Introduce the convenient coordinate components of ξ^i by $\xi^i = \vartheta^i{}_\mu \xi^\mu$, so

$$G = -\dot{\xi}^\mu \vartheta^i{}_\mu \pi_i^0 - \xi^\mu P_\mu,$$

$$P_\mu = \vartheta^i{}_\mu (\mathcal{H}_i + \vartheta^k{}_0 T^n{}_{ik} \pi_n^0) + \pi_i^0 \partial_\mu \vartheta^i{}_0 = \vartheta^i{}_\mu \mathcal{H}_i + \pi_i^0 \partial_\mu \vartheta^i{}_0.$$

Using the on shell relation $\partial_0 \vartheta^i{}_0 = u^i{}_0$, one obtains

$$P_0 = H_c + u^i{}_0 \pi_i^0 = H_T - \partial_\alpha D^\alpha,$$

$$P_\alpha = \mathcal{H}_\alpha + \pi_i^0 \partial_\alpha \vartheta^i{}_0 = \pi_i{}^\mu \partial_\alpha \vartheta^i{}_\mu - \partial_\beta (\pi_i{}^\beta \vartheta^i{}_\alpha).$$

This form of G correctly reproduces the local translations as a symmetry of the generic TG.

Generalization to $f(\mathbb{T})$ gravity

- ▶ Consider the formalism where the dynamical Hamiltonian is defined by the Legendre transform, which means that C_{ij} is included in \mathcal{H}_\perp . Then, the total/canonical Hamiltonians can be written in a form representing an isomorphic image of the TG formulas. In particular, the structure functions of the PB algebra in $f(\mathbb{T})$ gravity are identical to those of the PB algebra in TG. Hence, the Castellani procedure is practically identical to the one used in TG. As a result, the gauge generator in $f(\mathbb{T})$ gravity is found to be

$$\bar{G} \approx -\xi^i \pi_i^0 - \xi^i (\bar{\mathcal{H}}_i + \vartheta^k{}_0 T^k{}_{ik} \pi_i^0).$$

Here, the weak equality appears, however, one can show that the weak equality can be safely replaced by the strong one, so that

$$\bar{G} = -\xi^\mu \vartheta^\mu{}_\mu \pi_i^0 - \xi^\mu P_\mu,$$

$$P_0 = \bar{H}_c + u^i{}_0 \pi_i^0 = H_T - \partial_\alpha D^\alpha,$$

$$P_\alpha = \bar{\mathcal{H}}_\alpha + \pi_i^0 \partial_\alpha \vartheta^i{}_0 = \pi_i^\mu \partial_\alpha \vartheta^\mu{}_\mu - \partial_\beta (\pi_i^\beta \vartheta^i{}_\alpha) + \pi_\phi \partial_\alpha \phi.$$

One can show that the gauge generator \bar{G} produces the correct local translations when acting on the phase-space variables $(\vartheta^i{}_\mu, \pi_i^\mu)$. Moreover, a direct verification shows that its action on (ϕ, π_ϕ) is also correct. Hence, \bar{G} acts correctly on the whole phase space of $f(\mathbb{T})$ gravity.

- ▶ The gauge generator \bar{G} is constructed by assuming that π_i^0 is FC and using the PB algebra. The fact that \bar{G} is the true gauge generator of $f(\mathbb{T})$ gravity implies that the Hamiltonian constraints $(\bar{\mathcal{H}}_\perp, \bar{\mathcal{H}}_\alpha)$ must be FC, independently of the properties of other constraints, like π_ϕ, C_{ij} or χ_ϕ, χ_{ij} .

This is just as we expected from general principles, but it is a good check on everything to verify from the formalism.

Determining the multipliers (u_ϕ, u^{ij})

Preservation of the primary constraints π_ϕ and C_{mn} leads to conditions which either produce new constraints or determine some multipliers. These conditions can be written in the form

$$\chi_\phi : \quad u_{\perp\bar{j}} F^{\perp\bar{j}} + \frac{1}{2} u_{\bar{i}\bar{j}} F^{\bar{i}\bar{j}} \approx N F_\phi ,$$

$$\chi_{ij} : \quad F_{ij} \bar{u}_\phi + 2a_0 J \phi_{\bar{k}} \delta_{ij}^{\perp\bar{k}\bar{n}} u^m{}_n \approx X_{ij} ,$$

$$\bar{u}_\phi := N \partial_\perp \phi = u_\phi - N^\beta \partial_\beta \phi , \quad \phi_{\bar{k}} := \partial_{\bar{k}} \phi ,$$

$$X_{ij} := N \phi_{\bar{k}} \left[\phi^{-1} g^{\bar{k}\bar{m}} (n_i \hat{\pi}_{(\bar{m}\bar{j})} - n_j \hat{\pi}_{(\bar{m}\bar{i})}) + a_0 J \delta_{\bar{i}\bar{j}\bar{r}}^{\bar{k}\bar{m}\bar{n}} T^{\bar{r}}{}_{\bar{m}\bar{n}} \right] .$$

A 1+3 decomposition of χ_{ij} with $Z_{\bar{k}} := 2a_0 J \phi_{\bar{k}}$ gives

$$F_{\perp\bar{j}} \bar{u}_\phi - Z_{\bar{k}} u^{\bar{k}}{}_{\bar{j}} = X_{\perp\bar{j}} , \quad F_{\bar{i}\bar{j}} \bar{u}_\phi + Z_{\bar{k}} (\delta_{\bar{i}}^{\bar{k}} u^{\perp}{}_{\bar{j}} - \delta_{\bar{j}}^{\bar{k}} u^{\perp}{}_{\bar{i}}) = X_{\bar{i}\bar{j}} ,$$

Then, using $F_{\bar{k}} := \frac{1}{2} \varepsilon_{\bar{k}\bar{m}\bar{n}} F^{\bar{m}\bar{n}}$, $u_{\bar{k}} := \frac{1}{2} \varepsilon_{\bar{k}\bar{m}\bar{n}} u^{\bar{m}\bar{n}}$, the system of 7 equations for the 7 unknown multipliers $(\bar{u}_\phi, u_{\perp\bar{k}}, u_{\bar{k}})$ becomes

$$F^{\perp\bar{n}} u_{\perp\bar{n}} + F^{\bar{n}} u_{\bar{n}} \approx N F_\phi ,$$

$$F^{\perp\bar{m}} \bar{u}_\phi - Z_{\bar{k}} \varepsilon^{\bar{k}\bar{m}\bar{n}} u_{\bar{n}} = X^{\perp\bar{m}} ,$$

$$F^{\bar{m}} \bar{u}_\phi - Z_{\bar{k}} \varepsilon^{\bar{k}\bar{m}\bar{n}} u_{\perp\bar{n}} = X^{\bar{m}} .$$

For an extension to D spacetime dimensions, see Appendix F.

$$\phi_{\bar{k}} \neq 0$$

The contraction of the 2nd and 3rd equations with $\phi_{\bar{m}}$ yields

$$\phi_{\bar{m}} F^{\perp \bar{m}} \bar{u}_{\phi} = \phi_{\bar{m}} X^{\perp \bar{m}}, \quad \phi_{\bar{m}} F^{\bar{m}} \bar{u}_{\phi} = \phi_{\bar{m}} X^{\bar{m}}.$$

Two important consequences. *First, they generically determine \bar{u}_{ϕ} ,*

$$[(\phi_{\bar{m}} F^{\perp \bar{m}})^2 + (\phi_{\bar{m}} F^{\bar{m}})^2] \bar{u}_{\phi} = (\phi_{\bar{m}} F^{\perp \bar{m}}) \phi_{\bar{n}} X^{\perp \bar{n}} + (\phi_{\bar{m}} F^{\bar{m}}) \phi_{\bar{m}} X^{\bar{m}},$$

as long as $(\phi_{\bar{m}} F^{\perp \bar{m}})^2 + (\phi_{\bar{m}} F^{\bar{m}})^2 \neq 0$. **This is our main generic assumption;** it implies that at least one of the two terms $\phi_{\bar{m}} F^{\perp \bar{m}}$ and $\phi_{\bar{m}} F^{\bar{m}}$ does not vanish. *Second, they give a new secondary constraint,*

$$\chi := \phi_{\bar{n}} F^{\perp \bar{n}} (\phi_{\bar{m}} X^{\bar{m}}) - \phi_{\bar{n}} F^{\bar{n}} (\phi_{\bar{m}} X^{\perp \bar{m}})$$

A more detailed expression for χ is obtained using the identities

$$\begin{aligned} F_{\perp \bar{j}} &\equiv 2a_0 J \bar{V}_{\bar{j}}, & F_{\bar{i} \bar{j}} &\equiv -2a_0 J T^{\perp}_{\bar{i} \bar{j}}, & F^{\bar{k}} &\equiv -a_0 J \varepsilon^{\bar{k} \bar{m} \bar{n}} T_{\perp \bar{m} \bar{n}}, \\ X_{\perp \bar{j}} &\equiv N \phi^{-1} \phi^{\bar{m}} \hat{\pi}_{(\bar{m} \bar{j})}, & X_{\bar{i} \bar{j}} &\equiv a_0 b \phi_{\bar{k}} \delta_{\bar{i} \bar{j} \bar{r}}^{\bar{k} \bar{m} \bar{n}} T^{\bar{r}}_{\bar{m} \bar{n}}, & X^{\bar{i}} &\equiv a_0 b \phi_{\bar{k}} \varepsilon^{\bar{k} \bar{m} \bar{n}} T^{\bar{i}}_{\bar{m} \bar{n}}, \end{aligned}$$

$$\Rightarrow \quad \chi = a_0 b \varepsilon^{\bar{k} \bar{m} \bar{n}} \phi_{\bar{k}} \phi_{\bar{i}} \phi_{\bar{j}} \phi^{-1} (T_{\perp \bar{m} \bar{n}} \hat{\pi}^{(\bar{i} \bar{j})} + 2a_0 J \phi \bar{V}^{\bar{i}} T^{\bar{j}}_{\bar{m} \bar{n}}).$$

Remark

Here one can explicitly see that the $f(\mathbb{T})$ theory is indeed vulnerable to problems with nonlinear constraints leading to multipliers which can become unbounded for certain field values. This is an indication of a tachyonic propagating mode: in this case when $\phi_{\bar{m}} F^{\perp\bar{m}}$ or $\phi_{\bar{m}} F^{\bar{m}}$ approach zero \bar{u}_ϕ becomes unbounded—unless the right-hand sides also vanish.

This sort of think was noticed earlier in a teleparallel theory and also in the Poincaré gauge theory.

W.-H. Cheng, D.-C. Chern, and J. M. Nester, Canonical analysis of the one parameter teleparallel gravity, *Phys. Rev. D* **38** (1988) 2656–2658.

H. Chen, J. M. Nester and H.-J. Yo, Acausal PGT modes and the nonlinear constraint effect, *Acta Phys. Pol. B* **29** (1998) 961–970.

To examine how this affects the multipliers $u_{\perp \bar{k}}$ and $u_{\bar{k}}$, split them into components parallel to and orthogonal to $\phi_{\bar{k}}$,

$$\begin{aligned} u_{\perp \bar{k}} &= u_{\perp} \phi_{\bar{k}} + \hat{u}_{\perp \bar{k}}, & \hat{u}_{\perp \bar{k}} \phi_{\bar{k}} &= 0, \\ u_{\bar{k}} &= u \phi_{\bar{k}} + \hat{u}_{\bar{k}}, & \hat{u}_{\bar{k}} \phi_{\bar{k}} &= 0. \end{aligned}$$

Returning to the general conditions (0.3), note first that two equations contain only the orthogonal components $\hat{u}_{\bar{n}}$ and $\hat{u}_{\perp \bar{n}}$,

$$\begin{aligned} F^{\perp \bar{m}} \bar{u}_{\phi} - 2a_0 J \phi_{\bar{k}} \varepsilon^{\bar{k} \bar{m} \bar{n}} \hat{u}_{\bar{n}} &= X^{\perp \bar{m}}, \\ F^{\bar{m}} \bar{u}_{\phi} - 2a_0 J \phi_{\bar{k}} \varepsilon^{\bar{k} \bar{m} \bar{n}} \hat{u}_{\perp \bar{n}} &= X^{\bar{m}}. \end{aligned}$$

Then, substituting the solutions of these eqs for $\hat{u}_{\bar{n}}$ and $\hat{u}_{\perp \bar{n}}$ into the first condition yields one linear equation for the parallel components u_{\perp} and u ,

$$u_{\perp} (F^{\perp \bar{n}} \phi_{\bar{n}}) + u (F^{\bar{n}} \phi_{\bar{n}}) = N F_{\phi} - F^{\perp \bar{n}} \hat{u}_{\perp \bar{n}} - F^{\bar{n}} \hat{u}_{\bar{n}}.$$

The second equation for u and u_{\perp} is obtained from the term $\{C_{mn}, \chi'\}$ in the preservation condition for χ (see Appendix G):

$$\begin{aligned} -\partial_0 \chi' &= \{H_T, \chi'\} = \frac{1}{2} u^{mn} \{C_{mn}, \chi'\} + u \text{ independent terms} \\ &= u_{\perp} (\phi^{\bar{n}} \{C_{\perp \bar{n}}, \chi'\}) + u \left(\frac{1}{2} \varepsilon^{\bar{m} \bar{n} \bar{k}} \phi_{\bar{k}} \{C_{mn}, \chi'\} \right) + u \text{ independent terms.} \end{aligned}$$

The solutions of the system of linear equations (0.6) for u_{\perp} and u depend on the form of the determinant

$$D(x, x') := F^{\perp \bar{n}} \phi_{\bar{n}} \left(\frac{1}{2} \varepsilon^{\bar{i} \bar{j} \bar{k}} \phi_{\bar{k}} \{C_{\bar{i} \bar{j}}, \chi'\} \right) - (F^{\bar{n}} \phi_{\bar{n}}) (\phi^{\bar{j}} \{C_{\perp \bar{j}}, \chi'\}).$$

Suppose that $F^{\perp\bar{n}}\phi_{\bar{n}} \neq 0$. (The case $F^{\bar{n}}\phi_{\bar{n}} \neq 0$ is similar.) Then, (0.6a) is an equation that defines u_{\perp} in terms of u . Next, introduce the notation

$$u_{\perp}(\phi^{\bar{n}}\{C_{\perp\bar{n}}, \chi'\}) + u\left(\frac{1}{2}\varepsilon^{\bar{m}\bar{n}\bar{k}}\phi_{\bar{k}}\{C_{\bar{m}\bar{n}}, \chi'\}\right) =: g(x'),$$

multiply this relation by $F^{\perp\bar{p}}\phi_{\bar{p}}$,

$$u_{\perp}(F^{\perp\bar{p}}\phi_{\bar{p}})(\phi^{\bar{n}}\{C_{\perp\bar{n}}, \chi'\}) + u(F^{\perp\bar{p}}\phi_{\bar{p}})\left(\frac{1}{2}\varepsilon^{\bar{m}\bar{n}\bar{k}}\phi_{\bar{k}}\{C_{\bar{m}\bar{n}}, \chi'\}\right) = F^{\perp\bar{k}}\phi_{\bar{k}}g(x'),$$

insert the expression for u_{\perp} determined by (0.6a), and rearrange to get

$$u\left[F^{\perp\bar{p}}\phi_{\bar{p}}\left(\frac{1}{2}\varepsilon^{\bar{m}\bar{n}\bar{k}}\phi_{\bar{k}}\{C_{\bar{m}\bar{n}}, \chi'\}\right) - F^{\bar{p}}\phi_{\bar{p}}(\phi^{\bar{n}}\{C_{\perp\bar{n}}, \chi'\})\right] = \text{known terms.}$$

This equation for the last undetermined multiplier has the form

$$\boxed{u(x)D(x, x') = G(x')},$$

where $D(x, x')$ is defined in (0.7).

In view of the derivatives of the δ function buried in $D(x, x')$, see Appendix G, it is important to be mindful of the implicit integration over the variable x . Carrying out the integrations by parts (and then in the end dropping the prime) will lead to a relation of the form

$$A^\gamma \partial_\gamma u + \alpha u = G.$$

Remark: This is the first time that we have encountered a *differential equation* for a multiplier. It seems strange to us. How does this affect locality?

- (s1) *Generic scenario.* If the differential equation can be solved for u , all the multipliers are determined. Then the numbers of Lagrangian variables, first and second class constraints are, respectively, $N = 16 + 1$, $N_1(\pi_i^0, \mathcal{H}_i) = 8$, $N_2(\pi_\phi, C_{ij}, \chi) = 8$, then the number of d.o.f. is $N^* = 16 + 1 - 8 - 8/2 = 5$. Such a scenario could be realized if A^γ vanishes but $\alpha \neq 0$. In that special case the relation degenerates to a *linear algebra relation* for the final multiplier.
- (s2) If A^γ and α both vanish (highly unlikely we think), then G is a new secondary which must be preserved. That in turn could lead to further constraints with either the remaining multiplier being eventually determined or remaining undetermined. We do not see any way in principle to restrict the possible length of this constraint chain. If it is long enough there will be no d.o.f.
- (s3) We cannot exclude some other, albeit unlikely, possibilities. Thus, for instance, if $\partial_0 \chi$ identically vanishes, there would be only one condition for the multipliers u and u_\perp . Then, there would remain one undetermined multiplier, one degree of “remnant local Lorentz symmetry”. As a consequence, one combination of the components of (π_ϕ, C_{ij}) would be first class and would lead to a first class secondary χ , and six components of (π_ϕ, C_{ij}) would be second class. Then, $N_1 = 8 + 2 = 10$, $N_2 = 6$, and $N^* = 17 - 10 - 6/2 = 4$.

Most likely there are more than the 3 d.o.f. claimed by Ferraro and Guzmán (2018), and no—or at most 2—“remnant” local Lorentz symmetries, not 5.

Sector $\phi_{\bar{k}} = 0$

A field's phase space may have various sectors with distinct dynamics. One way this can happen is the subset of initial data with some special symmetry (e.g., spherical, axisymmetry, homogeneous). Another way is by restricting to the subset of fields where some quantities vanish.

Here we are focused on the ϕ -Lorentz sector with the primary constraints π_ϕ and C_{ij} . Preserving these constraints lead to certain conditions: (0.3).

Clearly, there is a special sector $\bar{\Gamma}$ of the whole phase space Γ , defined by

$$\phi_{\bar{k}} = 0.$$

This is a very important sector; it includes the homogeneous cosmologies where $\phi = \phi(t)$.

Note that there are different types of vanishing. One case is *instantaneous*, that is a quantity vanishes at $t = t_0$, but not at earlier or later instants. Such a case need not be pursued, as one can just adjust the initial time a little to avoid the vanishing. (However, one should then be concerned that the evolution stays regular as one approaches the critical time. One should instead focus on the cases where the system evolution stays on the subset where some quantity vanishes. Another complication that could happen but cannot be treated generally is where a quantity vanishes on a subset of the spatial hypersurface.

The content of this restriction is clarified by the following observations. The differential conditions $\partial_{\bar{k}}\phi = 0$ that define $\bar{\Gamma}$ do not eliminate ϕ as a degree of freedom, they only restrict the coordinate dependence of ϕ (invariance under spatial translations). Hence, they do not change the dimension of the phase space.

Additional information on this restriction comes from its dynamical preservation,

$$\partial_0\phi_{\bar{k}} := \{\phi_{\bar{k}}, H_T\} = \{\phi_{\bar{k}}, u'_\phi\pi'_\phi + C'_{ij}(u^{ij})'/2\} = u'_\phi\partial_{\bar{k}}\delta - u'_{\bar{k}}\bar{j}\phi_{\bar{j}} \approx 0,$$

By partial integration, one finds $\partial_{\bar{k}}u_\phi = 0$, which is just a consistent extension of the condition on ϕ to an analogous condition on $u_\phi = \partial_0\phi$. In the sector $\bar{\Gamma}$, the relation simplifies into $\bar{u}_\phi = u_\phi$ and the relations (0.3) reduce to

$$\begin{aligned} F^{\perp\bar{n}}u_{\perp\bar{n}} + F^{\bar{n}}u_{\bar{n}} &\approx NF_\phi, \\ F^{\perp\bar{m}}u_\phi &= 0, \\ F^{\bar{m}}u_\phi &= 0. \end{aligned}$$

In contrast to (0.3), the conditions (0.9) do not produce any additional constraint χ . The physical content of (0.9) is strongly influenced by the following theorem.

T1. If at least one of $F^{\perp\bar{m}}$, $F^{\bar{m}}$ is nonvanishing, then $u_\phi = 0$ and, as a consequence, $\partial_0\phi = \{\phi, H_T\} = \{\phi, u'_\phi\pi'_\phi\} = 0$. Combining this result with $\phi_{\bar{k}} = 0$, one finds that ϕ must be a true constant, $\phi = c$. As a consequence, the dynamical content of the Lagrangian field equation (0.2) takes the GR $_{\parallel}$ form, up to a cosmological constant term $V(c)/c$.

According to Eq. (0.1a), $\phi = c$ implies that \mathbb{T} is also a constant. Thus, any constant \mathbb{T} configuration allows the existence of nontrivial solutions provided they are also solutions of GR $_{\parallel}$ with a cosmological constant, which is a rather strong, nonperturbative restriction on the $f(\mathbb{T})$ dynamics in the “cosmological” sector $\phi_{\bar{k}} = 0$.

To complete our canonical analysis, let us now examine the corresponding number of d.o.f.. Theorem **T1** implies that the conditions (0.9) have just one physically relevant realization:

$$\begin{aligned} F_{\perp\bar{k}} &\equiv 2a_0 J \bar{V}_{\bar{k}} = 0, \\ F^{\bar{k}} &\equiv -a_0 J \varepsilon^{\bar{k}\bar{m}\bar{n}} T_{\perp\bar{m}\bar{n}} = 0, \\ \Rightarrow F_\phi &\equiv \frac{1}{2a_0\phi^2} P^2 + J(\mathbb{T} - \partial_\phi V) = 0. \end{aligned}$$

Since these conditions restrict the dimension of the phase space, one should impose their dynamical preservation.

Taking into account the relation $\{J, \mathcal{H}'_{mn}\} = 0$, see **(??)**₃, the preservation of $F_{\perp \bar{n}}$ becomes equivalent to (integration over x' implicit)

$$\partial_0 \bar{V}_{\bar{n}} = \{\bar{V}_{\bar{n}}, H_T\} = \{\bar{V}_{\bar{n}}, \mathcal{H}'_{ij}\} \frac{1}{2} (u^{ij})' + \{\bar{V}_{\bar{n}}, \check{H}_c\} \approx 0,$$

where we used the expression (0.6) for H_T . A direct inspection of the second term shows that it does not depend on the canonical multipliers. When the above relation, with interchanged x and x' , is combined with **(??)**₄, it yields

$$\begin{aligned} \partial_0 \bar{V}'_{\bar{n}} &= -\frac{1}{2} u^{ij} \{\mathcal{H}_{ij}, \bar{V}'_n\} + \beta'_{\bar{n}} \\ &\approx -(u^{\bar{i}\bar{j}} T_{\bar{j}\bar{n}} \delta + u^{\bar{i}}_n \bar{V}'_{\bar{i}} - u^{\perp \bar{j}} T_{\perp \bar{j}\bar{n}}) \delta - (e_{\bar{i}}^{\beta} e_{\bar{n}}^{\gamma})' [\partial'_{\beta} (u^i_{\gamma} \delta) - \partial'_{\gamma} (u^i_{\beta} \delta)] + \beta'_{\bar{n}}, \end{aligned}$$

where $\beta'_{\bar{n}} := -\{\check{H}_c, \bar{V}'_{\bar{n}}\}$. Integrating over x , and replacing x' by x in the final result, one obtains three differential conditions on the six multipliers u^{ij} ,

$$-(u^{\bar{i}\bar{j}}T_{\bar{j}\bar{i}\bar{n}} + u^{\bar{i}}_n \bar{V}_i - u^{\perp\bar{j}}T_{\perp\bar{j}\bar{n}}) + (e_i^\beta e_n^\gamma)(\partial_\beta u^i_\gamma - \partial_\gamma u^i_\beta) + \beta_{\bar{n}} \approx 0.$$

Similarly, the preservation of $F^{\bar{n}}$ is equivalent to

$$\partial_0 T_{\perp\bar{m}\bar{n}} \approx \{T_{\perp\bar{m}\bar{n}}, \mathcal{H}'_{ij}\} \frac{1}{2} (u^{ij})' + \beta_{\perp\bar{m}\bar{n}} \approx 0.$$

Then, relation $(??)_2$ implies

$$\partial_0 T'_{\perp\bar{m}\bar{n}} = - \left[(u^{\perp\bar{j}}T_{\bar{j}\bar{m}\bar{n}} + u^{\bar{i}}_m T_{\perp\bar{i}\bar{n}} - u^{\bar{i}}_n T_{\perp\bar{i}\bar{m}}) \delta - n'_i (e_{\bar{m}}^\beta e_{\bar{n}}^\gamma)' [\partial'_\beta (u^i_\gamma \delta) - \partial'_\gamma (u^i_\beta \delta)] \right] +$$

and consequently,

$$-u^{\perp\bar{j}}T_{\bar{j}\bar{m}\bar{n}} - u^{\bar{i}}_m T_{\perp\bar{i}\bar{n}} + u^{\bar{i}}_n T_{\perp\bar{i}\bar{m}} + n_i (e_{\bar{m}}^\beta e_{\bar{n}}^\gamma) (\partial_\beta u^i_\gamma - \partial_\gamma u^i_\beta) + \beta_{\perp\bar{m}\bar{n}} \approx 0.$$

Here, we have a set of three conditions on the six multipliers $(u^{\bar{i}\bar{j}}, u^{\perp\bar{j}})$.

Finally, the preservation of F_ϕ takes the form

$$\partial_0 F_\phi = \{F_\phi, H_T\} = \{F_\phi, u'_\phi \pi'_\phi\} + \text{more} = u_\phi (\partial_\phi F_\phi) + \text{more} \approx 0.$$

Relations (0.11a) and (0.11a) are differential equations for the canonical multipliers u^{ij} , whereas the condition (0.11a) determines u_ϕ , provided $\partial_\phi F_\phi \neq 0$.

(s4) In the *generic scenario*, relations (0.11) determine the multipliers (u_ϕ, u^{ij}) . Then, in the phase space $\bar{\Gamma}$ with 17 Lagrangian variables (ϑ^i_μ, ϕ) , we have

$3+3+1=7$ new constraints (0.10),

seven preservation conditions (0.11), and seven determined multipliers (u^{ij}, u_ϕ) .

Since the seven primary constraints (π_ϕ, C_{ij}) and the seven new constraints (0.10) are second class, and $N_1 = 8$, **the number of physical d.o.f. is $N^* = 17 - 8 - 7 = 2$, the same as in GR $_{||}$.**

This result was to be expected, since, as we noted at the end of **T1**, in this case the Lagrangian equations reduce to those of GR with a cosmological constant. However, we find it instructive, and a good consistency check, to obtain this result within the Hamiltonian analysis.

Summary and discussion

- ▶ We performed a detailed Hamiltonian analysis of $f(\mathbb{T})$ gravity, focusing on local Lorentz invariance, the number of physical d.o.f., and the issue of nonlinear constraint effects. Our main results can be summarized as follows.
- ▶ (r1) The central role of the Lorentz constraint C_{ij} with respect to the status of local Lorentz invariance can be seen already at an early stage of the canonical analysis. Namely, by showing that $\{C_{ij}, C_{kl}\}$ does not vanish weakly, which means that C_{ij} is not first class, one can conclude that local Lorentz invariance is broken.
- ▶ (r2) To classify all the constraints and calculate the number of physical d.o.f., we found it convenient to first prove the first-class nature of the ADM components of the canonical Hamiltonian. This significantly simplifies further analysis and, as an “aside” but expected result, it implies the diffeomorphism invariance of $f(\mathbb{T})$ gravity.

- ▶ (r3) The classification of the remaining constraints is based on the preservation conditions of the primary constraints (π_ϕ, C_{ij}) , interpreted as seven conditions on the seven multipliers (u_ϕ, u^{ij}) . Then we analyzed which of these multipliers are determined (that is, associated to second-class constraints) and what happens with secondary constraints, if they exist. We found that generically, the number of physical d.o.f. is $N^* = 5$. Note also that in the special case $\partial_{\bar{k}}\phi = 0$, $f(\mathbb{T})$ gravity reduces to GR with a cosmological constant, with $N^* = 2$.
- ▶ (r4) We confirmed that $f(\mathbb{T})$ gravity is indeed vulnerable to the effects associated with nonlinear second class constraints. When the dynamical variables evolve toward values such that certain quantities approach zero, certain canonical multipliers can diverge—signaling an associated anomalous propagation. Such behavior can be an indication of a fatal problem.

Comparison with earlier $f(\mathbb{T})$ Hamiltonian investigations

To gain a deeper insight into our results, we compare them to those of Li, Miao, Miao (2011) and Ferraro and Guzmán (2018)

1. The basic results of Li et al. in $D = 4$ are presented in Section 4 of LMM (2011). Their Eqs. (25)–(28), representing the PB algebra involving the set of the ϕ -Lorentz primary constraints (π_ϕ, C_{ij}) and the canonical Hamiltonian H_0 , are in complete agreement with our findings. In particular, their PB (25), with G^{ab} given in the first line of the next page, is identical to our result for $\{C_{ij}, C_{mn}\}$. On the other hand, it should be contrasted with the Lorentz PB algebra closure found in Eq. (70)₁ of F&G (2018). We did not find any comment by Ferraro and Guzmán on this disagreement, although it is of essential importance for the Lorentz invariance and the counting of d.o.f.. Next, Li et al. continue with the analysis of the 3 equations (29) by interpreting them as $1 + 6 + 1 = 8$ conditions for the 7 multipliers. In addition to the last two equations that we considered (7 conditions for 7 multipliers), they included here also the preservation condition for the canonical Hamiltonian H_0 , equal to our \check{H}_c . In our approach, we gave a completely separate discussion of the preservation of \check{H}_c . Namely, in sections 4 and 5, we showed that a suitably modified canonical Hamiltonian $\bar{H}_c = \check{H} + (1/2)u^{mn}C_{mn} + u_\phi\pi_\phi$ is first class, which implies its preservation. Hence, Eq. (29)₁ is not really needed, it is just a consequence of the last two

Without knowing that, the authors continue by writing the 8 conditions (29) in the form of a homogeneous matrix equation with an 8×8 antisymmetric matrix M having a vanishing determinant and rank 6. The condition $\det M = 0$ is written as a new constraint $\pi_1 := \sqrt{\det M} \approx 0$, whose preservation $\partial_0 \pi_1 \approx 0$ yields a new condition on the multipliers. Alternatively, disregarding the redundant Eq.(29)₁, one is left with $6 + 1 = 7$ conditions for 7 multipliers. As we showed in subsection 6.1, the 2nd eqn gives (generically) 5 conditions on multipliers plus a secondary constraint χ , the preservation of which produces one more condition on the multipliers, $\partial_0 \chi \approx 0$. Thus, the 7 equations (29)₂ and (29)₃ could be written as a homogeneous matrix equation, with a 7×7 matrix of rank 6. This confirms that Eq.(29)₁ is indeed superfluous. Using it does not do any harm, but it does complicate the analysis. Although both approaches in the *generic* scenario predict the same number of d.o.f., $N^* = 5$, our formalism is more explicit and practical. At the end of section 5, Li et al. discuss the d.o.f. for a D -dimensional spacetime. The result of our analysis in Appendix F, $(D - 1)$ d.o.f., agrees with their finding in the Lorentz ϕ sector. As a final remark, in discussing the results obtained from the second equation in D dimensions, the authors write: “One can check that in four dimensions the constraint derived from the second equation of eq.(29) and square root of the determinant of M eq.(36) are exactly the same.” This means that our χ coincides with their π_1 . That means that our secondary constraint result is “exactly the same” as theirs. Also, in their appendix, they find, just as we did, a first order differential equation for the last canonical multiplier.

2. As mentioned, one of the main errors in Ferraro and Guzmán (2018) is their claim that the PB algebra of the constraints $G_{ab}^{(1)}$ in Eq. (70)₁ closes just like the ordinary Lorentz algebra, which is in contradiction to our result [and Li et al. Eq. (25)]. This error seriously affects their analysis, leading them to claim that 5 of the 6 constraints $G_{ab}^{(1)}$ are first class, not second class.

There are other errors; we comment here on only two. We begin by noting that the last equality in their (65) implies $F_\phi = 0$. Indeed, as shown in our Appendix B, the Hamiltonian transcript of $E(\mathbb{T} - \partial_\phi V)$ weakly vanishes. Then, since F_ϕ introduced in their Eq. (64) does not vanish, the last equality in (65) cannot be correct.

Moreover, they calculated the preservation condition for $G_{ab}^{(1)}$ in their Eq. (81)₂. By comparison to our Appendix C, their result is recognized just as a fraction of the complete result, associated to our coefficient B_4 .

Ferraro and Guzmán have published several follow-up works, which have already attracted considerable attention. As they were based on the unsound foundation, they are not reliable guides.

Conclusion

- ▶ In this paper we presented a detailed analysis of the puzzling $f(\mathbb{T})$ Hamiltonian/constraint/d.o.f. issues. This could be used as a solid foundation for certain future investigations into the nature of this curious theory.

Appendix C: The preservation of C_{ij}

To calculate the preservation condition $\partial_0 C_{ij} \approx 0$, we start from the relation

$$\chi_{ij} := \{C_{ij}, H_T\} = \underbrace{\{C_{ij}, N\mathcal{H}_\perp\}}_{B_1} + \underbrace{\{C_{ij}, N^\beta \mathcal{H}_\beta\}}_{B_2} + \underbrace{\{C_{ij}, (1/2)u^{mn} C_{mn}\}}_{B_3} + \underbrace{\{C_{ij}, u_\phi \pi_\phi\}}_{B_4}$$

$$\begin{aligned} B_1 &= -\frac{1}{2}\partial_\alpha [N(n_i C_{jk} - n_j C_{ik})e^{\bar{k}\alpha} \delta] \\ &\quad -\phi^{-1}\partial_\alpha \phi N(n_i \hat{\pi}_{(j\bar{k})} - n_j \hat{\pi}_{(i\bar{k})})e^{\bar{k}\alpha} \delta - a_0(\partial_\alpha \phi)\varepsilon_{ijmk}^{0\alpha\beta\gamma} T^m{}_{\beta\gamma} N n^k \delta, \\ B_2 &= \partial_\beta (N^\beta C_{ij} \delta) - (N^\beta \partial_\beta \phi) a_0 B_{ij} \delta, \\ B_3 &= u^{mn}(g_{jm} C_{in} - g_{im} C_{jn})\delta + a_0 u^{mn} \partial_\alpha \phi (g_{j\bar{n}} B_{im}^{0\alpha} - g_{i\bar{n}} B_{jm}^{0\alpha})\delta, \\ B_4 &= a_0 B_{ij} u_\phi \delta = a_0 B_{ij} (N \partial_\perp \phi + N^\beta \partial_\beta \phi). \end{aligned} \quad (0.1)$$

Then, transition to the weak equality yields

$$\begin{aligned} B_1 &\approx -\phi^{-1}\partial_\alpha \phi N(n_i \hat{\pi}_{(j\bar{k})} - n_j \hat{\pi}_{(i\bar{k})})e^{\bar{k}\alpha} \delta - a_0(\partial_\alpha \phi)\varepsilon_{ijmk}^{0\alpha\beta\gamma} T^m{}_{\beta\gamma} N n^k \delta \\ B_3 &\approx a_0 u^{mn} \partial_\alpha \phi (g_{j\bar{n}} B_{im}^{0\alpha} - g_{i\bar{n}} B_{jm}^{0\alpha})\delta, \quad B_4 + B_2 \approx a_0 B_{ij} N \partial_\perp \phi. \end{aligned} \quad (0.2)$$

To get χ_{ij} transform the 2nd term in B_1 and all of B_3 with the help of

$$\vartheta^k{}_\alpha \varepsilon_{ijmr}^{\alpha 0\beta\gamma} T^m{}_{\beta\gamma} N n^r = -b \delta_{\bar{r}\bar{j}}^{\bar{k}\bar{m}\bar{n}} T^{\bar{r}}{}_{\bar{m}\bar{n}}, \quad \vartheta^k{}_\alpha u^{mn} (g_{j\bar{n}} B_{im}^{0\alpha} - g_{i\bar{n}} B_{jm}^{0\alpha}) = -2J \delta_{imj}^{\perp kn}$$

Appendix D

In this appendix, we examine what one gets if one tries to directly construct the $f(\mathbb{T})$ theory Hamiltonian. The $f(\mathbb{T})$ theory has the Lagrangian

$$\tilde{\mathcal{L}}_{fT} = \vartheta \mathcal{L}_{fT} = \vartheta f(\mathbb{T}), \quad (1.1)$$

where \mathbb{T} is the GR_{\parallel} expression, displayed in (2.4).

The conjugate momenta come from

$$\pi_i{}^\mu := \frac{\partial[\vartheta f(\mathbb{T})]}{\partial \dot{\vartheta}^i{}_\mu} = f'(\mathbb{T}) \vartheta \mathcal{H}_i{}^{0\mu}. \quad (1.2)$$

Once again, one finds the *sure* primary constraints

$$\pi_i{}^0 \approx 0. \quad (1.3)$$

We find it convenient to represent the parallel momenta in a suggestive form

$$\hat{\pi}_{i\bar{k}} = \Phi J \mathcal{H}_{i\perp\bar{k}}, \quad \Phi := f'(\mathbb{T}). \quad (1.4)$$

The momenta $\pi_i{}^{\bar{k}}$ can again be split into irreducible components to give

$$\begin{aligned} p_{\perp\bar{k}} &:= \hat{\pi}_{\perp\bar{k}}/J = -2\Phi T^{\bar{m}}{}_{\bar{m}\bar{k}}, \\ {}^A p_{\bar{i}\bar{k}} &:= {}^A \hat{\pi}_{\bar{i}\bar{k}}/J = \Phi T_{\perp\bar{i}\bar{k}}, \end{aligned} \quad (1.5)$$

$$\begin{aligned} p^{\bar{m}}{}_{\bar{m}} &:= \hat{\pi}^{\bar{m}}{}_{\bar{m}}/J = -4\Phi T^{\bar{m}}{}_{\perp\bar{m}}, \\ {}^T p_{\bar{i}\bar{k}} &:= {}^T \hat{\pi}_{\bar{i}\bar{k}}/J = 2\Phi {}^T T_{\bar{i}\perp\bar{k}}. \end{aligned} \quad (1.6)$$

Let us now combine the first two relations in the form

$$p_{ik} := \frac{1}{2J}(\pi_{i\bar{k}} - \pi_{k\bar{i}}) = \Phi \hat{F}_{ik}, \quad (1.7a)$$

$$\hat{F}_{ik} := (T^{\perp}_{i\bar{k}} - n_i T^{\bar{m}}_{\bar{m}\bar{k}} + n_k T^{\bar{m}}_{\bar{m}\bar{i}}), \quad (1.7b)$$

where both p_{ik} and \hat{F}_{ik} are antisymmetric objects. From this, it follows that

$$\hat{F} \cdot p := \hat{F}^{ik} p_{ik} = \Phi \hat{F}^2, \quad \hat{F}^2 := \hat{F} \cdot \hat{F} := \hat{F}^{ik} \hat{F}_{ik}. \quad (1.8)$$

Clearly, vanishing \hat{F}^2 is a special case. Let us put this case aside for separate investigation, and consider the *generic* case where \hat{F}^2 does not vanish *anywhere*. (One could also make a more complicated, “less covariant” analysis by considering the vanishing of $\hat{F}_{\perp\bar{k}}$ and $\hat{F}_{i\bar{k}}$ separately.) Then, we find from (1.8) the component of (1.7a) along \hat{F}_{ik} ,

$$\Phi \equiv f'(\mathbb{T}) = \frac{\hat{F} \cdot p}{\hat{F}^2}. \quad (1.9)$$

Using (1.9), one can invert the relations (1.6) for some of the “velocities”:

$$T^{\bar{m}}_{\perp\bar{m}} = -\frac{p^{\bar{m}}_{\bar{m}}}{4\Phi}, \quad T^T_{T_{i\perp\bar{k}}} = \frac{T p_{i\bar{k}}}{2\Phi}. \quad (1.10)$$

Furthermore, assuming $f''(\mathbb{T}) \neq 0$, by the implicit function theorem, the relation (1.9) can be inverted to give

$$\mathbb{T} = (f')^{-1}(\Phi). \quad (1.11)$$

With this relation, one can find the “missing” “antisymmetric velocity”–momentum relation for the one velocity component along F_{ik} from

$$\begin{aligned} \mathbb{T} &\simeq \frac{1}{2} \text{velocity}^2 + (\text{anti-sym velocity comp})^1 + \bar{\mathbb{T}} \\ &= \frac{1}{2\Phi^2} P^2 + \hat{F}^{ik} A T_{i\perp k} + \bar{\mathbb{T}}, \end{aligned} \quad (1.12)$$

where $\bar{\mathbb{T}}$ and P^2 are defined as in (3.9). No other components of the “antisymmetric velocity” can be inverted for momenta; instead, the other five components of (1.7a) are *new primary constraints*, which can be written as

$$\hat{C}_{ik} := p_{ik} - \Phi \hat{F}_{ik} \approx 0, \quad \hat{C}_{ik} \hat{F}^{ik} = 0. \quad (1.13)$$

It is interesting to point out certain structural similarities of the above results to those obtained in the $\phi\mathbb{T}$ formalism of section ???: first, the expression (1.13) is an analogue of the extra primary constraint (0.3), and second, the relation (1.12) is a counterpart of (??).

Now, one can construct the Hamiltonian. The *total* Hamiltonian has the form

$$\mathcal{H}_T = \mathcal{H}_c + u^i \pi_i^0 + \frac{1}{2} \hat{u}^{ik} \hat{C}_{ik}, \quad (1.14)$$

including 4+5 *primary* constraints with the canonical multipliers. The explicit form of \mathcal{H}_c can be found by following a close analogy to the procedure described in the main text, but the alternative $\phi\mathbb{T}$ formalism seems to be much more practical. Nevertheless, we want to stress that one could develop a complete Hamiltonian analysis based on (1.14). Unlike the case of the six Lorentz generators of GR_{\parallel} , one will now find that the Poisson brackets algebra among the five primary constraints \hat{C}_{ik} does not close, which is related to the fact that the Lorentz Lie algebra does not have a five-dimensional Lie subalgebra. From the analysis of the $\phi\mathbb{T}$ formulation discussed in detail in the text, we can infer that the five constraints (1.13) will be second class. If one goes further, one will find that the preservation of these five constraints will generically lead to four conditions on the five multipliers \hat{u}^{ik} plus one secondary constraint χ . The preservation of the latter will, generically, yield a first order differential equation for the last multiplier. Generically, the number of physical d.o.f. is then

$$N^* = N - N_1 - N_2/2 = 16 - 8 - 6/2 = 5 = 2 + 3. \quad (1.15)$$

Solving for the multipliers in dimension D

This is an alternative analysis to the one in Section ??, it is hardly more complicated and extends the result to D spacetime dimensions.

The equations to be solved are displayed in (0.1a) and (0.3):

$$F^{\perp\bar{j}}u_{\perp\bar{j}} - \frac{1}{2}F_{\bar{i}\bar{j}}u^{\bar{i}\bar{j}} + NF^{\phi} \approx 0, \quad (1.1)$$

$$F^{\perp\bar{j}}\bar{u}_{\phi} - Z_{\bar{k}}\delta_{\bar{l}\bar{m}}^{\bar{k}\bar{j}}\frac{1}{2}u^{\bar{l}\bar{m}} \approx X^{\perp\bar{j}}, \quad (1.2)$$

$$F_{\bar{i}\bar{j}}\bar{u}_{\phi} + Z_{\bar{k}}\delta_{\bar{i}\bar{j}}^{\bar{k}\bar{l}}u_{\perp\bar{l}} \approx X_{\bar{i}\bar{j}}. \quad (1.3)$$

These equations have, respectively, 1, $D - 1$, $(D - 1)(D - 2)/2$ components. The first relation gives *one* restriction on u^{ij} , let us set it aside for now. The component of the second relation projected along $\phi_{\bar{j}}$ is

$$(\phi_{\bar{j}}F^{\perp\bar{j}})\bar{u}_{\phi} \approx \phi_{\bar{j}}X^{\perp\bar{j}}. \quad (1.4)$$

Generically (i.e., when $\phi_{\bar{j}}F^{\perp\bar{j}} \neq 0$)

The unknown “velocity” multipliers u^{ij} can be split into components along and orthogonal to $\phi_{\bar{k}}$:

$$u_{\perp\bar{l}} = u_{\perp}\phi_{\bar{l}} + \hat{u}_{\perp\bar{l}}, \quad \hat{u}_{\perp\bar{l}}\phi^{\bar{l}} = 0, \quad (1.5)$$

$$u^{\bar{m}\bar{n}} = (u^{\bar{m}}\phi^{\bar{n}} - u^{\bar{n}}\phi^{\bar{m}}) + \hat{u}^{\bar{m}\bar{n}}, \quad u^{\bar{m}}\phi_{\bar{m}} = 0, \quad \hat{u}^{\bar{m}\bar{n}}\phi_{\bar{n}} = 0, \quad (1.6)$$

having, respectively, $1 + (D - 2) = D - 1$ and $(D - 2) + (D - 2)(D - 3)/2 = (D - 1)(D - 2)/2$ components.

Using this splitting and (1.4), the remaining part of (1.2) and (1.3) can be, respectively, rearranged into

$$(\phi_{\bar{m}} F^{\perp \bar{m}} Z_{\bar{k}} \phi^{\bar{k}}) u^{\bar{j}} \equiv -(\phi_{\bar{l}} F^{\perp \bar{l}}) Z_{\bar{k}} u^{\bar{k} \bar{j}} \approx \phi_{\bar{m}} [F^{\perp \bar{m}} X^{\perp \bar{j}} - F^{\perp \bar{j}} X^{\perp \bar{m}}], \quad (1.7)$$

$$(\phi_{\bar{m}} F^{\perp \bar{m}}) Z_{\bar{k}} \delta_{\bar{i} \bar{j}}^{\bar{k} \bar{l}} \hat{u}_{\perp \bar{l}} \approx \phi_{\bar{m}} [F^{\perp \bar{m}} X_{\bar{i} \bar{j}} - F_{\bar{i} \bar{j}} X^{\perp \bar{m}}]. \quad (1.8)$$

Generically, (1.7) is $D - 2$ equations which can be solved for the $D - 2$ components of $u^{\bar{j}}$. Contracting (1.8) with $\phi^{\bar{i}}$ yields

$$(\phi_{\bar{m}} F^{\perp \bar{m}}) (\phi^{\bar{i}} Z_{\bar{i}}) \hat{u}_{\perp \bar{j}} \approx \phi_{\bar{m}} \phi^{\bar{i}} [F^{\perp \bar{m}} X_{\bar{i} \bar{j}} - F_{\bar{i} \bar{j}} X^{\perp \bar{m}}], \quad (1.9)$$

which (generically) can be solved for the $D - 2$ components of $\hat{u}_{\perp \bar{j}}$. The remaining components of (1.8) orthogonal to $\phi_{\bar{k}}$ are $(D - 1)(D - 2)/2 - (D - 2) = (D - 2)(D - 3)/2$ *secondary* constraints,

$$\bar{\chi}_{\bar{r} \bar{s}} := \delta_{\bar{r} \bar{s} \bar{l}}^{\bar{k} \bar{i} \bar{j}} \phi^{\bar{l}} \phi_{\bar{k}} \phi_{\bar{m}} [F^{\perp \bar{m}} X_{\bar{i} \bar{j}} - F_{\bar{i} \bar{j}} X^{\perp \bar{m}}]. \quad (1.10)$$

Note the appearance of the projection operator

$$P_{\bar{r} \bar{s}}^{\bar{i} \bar{j}} := \delta_{\bar{r} \bar{s} \bar{l}}^{\bar{k} \bar{i} \bar{j}} \phi^{\bar{l}} \phi_{\bar{k}}, \quad (1.11)$$

which projects antisymmetric quantities (multiplied by a factor of $\phi_{\bar{k}} \phi^{\bar{k}}$) onto the subspace orthogonal to $\phi_{\bar{k}}$.

The preservation of the secondary constraint (1.10) will, upon introducing the values of the known quantities, yield a relation linear in the as-yet-undetermined u_{\perp} , $\hat{u}^{\bar{u}\bar{v}}$:

$$\begin{aligned}
0 = -\partial_0 \chi'_{\bar{r}\bar{s}} &= \{H_T, \chi'_{\bar{r}\bar{s}}\} \approx \frac{1}{2} u^{ij} \{C_{ij}, \chi'_{\bar{r}\bar{s}}\} + \text{known terms} \\
&= u^{\perp\bar{j}} \{C_{\perp\bar{j}}, \chi'_{\bar{r}\bar{s}}\} + \frac{1}{2} u^{\bar{i}\bar{j}} \{C_{\bar{i}\bar{j}}, \chi'_{\bar{r}\bar{s}}\} + \text{known terms} \\
&= u^{\perp} \phi^{\bar{j}} \{C_{\perp\bar{j}}, \chi'_{\bar{r}\bar{s}}\} + \frac{1}{4} (\phi_{\bar{k}} \phi^{\bar{k}})^{-1} P_{\bar{u}\bar{v}}^{\bar{i}\bar{j}} \hat{u}^{\bar{u}\bar{v}} \{C_{\bar{i}\bar{j}}, \chi'_{\bar{r}\bar{s}}\} + \text{known terms}
\end{aligned}$$

where r, s, u, v effectively range over the directions orthogonal to $\phi_{\bar{k}}$.

A similar splitting of (1.1) gives

$$\begin{aligned}
0 &\approx (\phi_{\bar{k}} \phi^{\bar{k}}) [F^{\perp\bar{j}} u_{\perp\bar{j}} - \frac{1}{2} F_{\bar{i}\bar{j}} u^{\bar{i}\bar{j}} + N F^{\phi}] \\
&= (\phi_{\bar{k}} \phi^{\bar{k}}) F^{\perp\bar{j}} u_{\perp\bar{j}} - \frac{1}{4} F_{\bar{i}\bar{j}} P_{\bar{u}\bar{v}}^{\bar{i}\bar{j}} \hat{u}^{\bar{u}\bar{v}} + \text{known terms.} \quad (1.13)
\end{aligned}$$

Rearranging, this gives u_{\perp} from

$$\phi_{\bar{j}} F^{\perp\bar{j}} u_{\perp} = (\phi_{\bar{k}} \phi^{\bar{k}})^{-1} \frac{1}{4} F_{\bar{i}\bar{j}} P_{\bar{u}\bar{v}}^{\bar{i}\bar{j}} \hat{u}^{\bar{u}\bar{v}} + \text{known terms.} \quad (1.14)$$

Inserting this into (1.13) leads to $(D-2)(D-3)/2$ linear relations for the remaining $(D-2)(D-3)/2$ unknowns $\hat{u}^{\bar{u}\bar{v}}$,

$$\hat{u}^{\bar{u}\bar{v}} P_{\bar{u}\bar{v}}^{\bar{i}\bar{j}} \left[F^{\perp}_{\bar{k}} \phi^{\bar{k}} \{C_{\bar{i}\bar{j}}, \chi'_{\bar{r}\bar{s}}\} - F_{\bar{i}\bar{j}} \phi^{\bar{k}} \{C_{\perp\bar{k}}, \chi'_{\bar{r}\bar{s}}\} \right] = \text{known terms.} \quad (1.15)$$

This equation for the remaining undetermined multipliers has the form

$$\hat{u}^{\bar{u}\bar{v}}(x)D_{\bar{u}\bar{v}\bar{r}\bar{s}}(x, x') = G_{\bar{r}\bar{s}}(x'). \quad (1.16)$$

When one calculates the Poisson brackets $\{C_{ij}, \chi'_{\bar{r}\bar{s}}\}$, one will get, in general, both terms proportional to the δ function and to its derivative. In view of the derivatives of the δ function buried in $D(x, x')$, it is important to be mindful of the implicit integration over the variable x . Carrying out the integrations by parts (and then, in the end, dropping the prime) will lead to a relation of the form

$$A_{\bar{u}\bar{v}\bar{r}\bar{s}}^{\gamma}\partial_{\gamma}\hat{u}^{\bar{u}\bar{v}} + \alpha_{\bar{u}\bar{v}\bar{r}\bar{s}}\hat{u}^{\bar{u}\bar{v}} = G_{\bar{r}\bar{s}}. \quad (1.17)$$

Thus, we get *generically a system of first-order linear differential equations* for the multipliers $\hat{u}^{\bar{u}\bar{v}}$, the solutions to such a system will thus have a certain degree of nonlocality, in comparison with the solutions of algebraic equations. The explicit functional forms for $A_{\bar{u}\bar{v}\bar{r}\bar{s}}^{\gamma}$ and $\alpha_{\bar{u}\bar{v}\bar{r}\bar{s}}$ in (1.17) can be straightforwardly obtained from the explicit form of $\{C_{ij}, \chi'_{\bar{r}\bar{s}}\}$. Several scenarios are possible. One can determine all the “missing” multipliers if this linear relation determines the $\hat{u}^{\bar{u}\bar{v}}$. Otherwise, some components of this relation may give some additional constraints, which should then be preserved. The chain of constraints could, in principle, go on for several steps before terminating. We cannot exclude the possibility that, in the end, some components of $\hat{u}^{\bar{u}\bar{v}}$ may remain undetermined, so that the solutions have some gauge freedom. However, we think that these possibilities are quite unlikely.

Generically, the constraints π_ϕ , C_{ij} , $\bar{\chi}_{\bar{r}\bar{s}}$ are $1 + D(D - 1)/2 + (D - 2)(D - 3)/2 = (D - 1)(D - 2) + 2$ second class constraints, and the number of d.o.f. in the ϕ -Lorentz sector is

$$D(D - 1)/2 + 1 - \frac{1}{2}[(D - 1)(D - 2) + 2] = D - 1. \quad (1.18)$$

For $D = 4$, this gives 3 d.o.f. beyond the metric. This is what we found in the main text, and exactly agrees with the claim of [?]. For $D > 4$ also, the analysis presented here leads to the same number of constraints as presented in that work, however, the formulas and the analysis appearing here are more detailed and simpler.

Although the relations presented here seem much more tractable than those in [?], explicitly verifying that the $\chi_{\bar{r}\bar{s}}$ are truly second class and their preservation leads to all the missing multipliers is not so easy. So we cannot yet exclude other possibilities, including the unlikely extreme case that the $\chi_{\bar{r}\bar{s}}$ are identically preserved. Then, they would be first class and $(D - 2)(D - 3)/2$ of the C_{ij} would also be first class. In this case, the Lorentz sector would have $(D - 2)(D - 3)$ first class constraints and $1 + 1 + 2(D - 2) = 2(D - 1)$ second class constraints. There are other unlikely possibilities. In any case, we can be sure that there are at least $2(D - 1)$ second class constraints and not $D(D - 1)/2 - 1$ first class, unlike the claims of [?].

Furthermore, there are indeed (as we had conjectured) some possibilities for problematical nonlinear constraint effects. Fixing the multipliers in the second class case requires $\phi_{\bar{k}} F^{\perp \bar{k}} \neq 0$ and a nondegeneracy of $D_{\bar{u}\bar{v}\bar{r}\bar{s}}$. The dynamics is prone to catastrophic behavior if these quantities degenerate somewhere. However, if ϕ is nonconstant and yet vanishes asymptotically at infinity, *it must have critical points somewhere*, so $\phi_{\bar{k}}$ can be expected to vanish at some points. Thus, indeed, there is good reason to be concerned about the effects of the changing of the rank of the constraint Poisson bracket matrix.

Appendix G

We shall focus here on the part $\{C_{ij}, \chi'\}$ of the complete expression $-\partial_0\chi$. The calculation will be organized in several simple tasks. Start by rewriting C_{ij} in the form

$$C_{ij} = \mathcal{H}_{ij} + a_0\phi B_{ij}, \quad \mathcal{H}_{ij} = \pi_{i\bar{j}} - \pi_{j\bar{i}}, \quad B_{ij} = \partial_\alpha B_{ij}^{0\alpha}. \quad (2.1)$$

In order to explore the dynamical content of χ , it is suitable to rewrite it in the form:

$$\begin{aligned} \chi &= a_0 b \phi^{-1} w_1 (w_2 + 2a_0 J \phi w_3), \\ w_1 &= \varepsilon^{\bar{k}\bar{m}\bar{n}} \phi_{\bar{k}} \phi_{\bar{r}} \phi_{\bar{s}}, \\ w_2 &:= T_{\perp\bar{m}\bar{n}} \hat{\pi}^{(\bar{r}\bar{s})}, \\ w_3 &:= \bar{V}^{\bar{r}} T^{\bar{s}}_{\bar{m}\bar{n}}, \end{aligned} \quad (2.2)$$

see Section **??**. The factors $f = (b, \phi, J)$ are singled out since $\{C_{ij}, f\} = 0$, see **(??)**. The indices of w_n can be reconstructed by $w_1 \rightarrow w_1^{\bar{m}\bar{n}}_{\bar{r}\bar{s}}$, $w_2 \rightarrow w_2^{\bar{r}\bar{s}}_{\bar{m}\bar{n}}$, and similarly for w_3 .

Step 1. We begin by calculating the terms $W_n := \{C_{ij}, w'_n\}$, using the formulas

$$\begin{aligned}
W_1 &= \{\mathcal{H}_{ij}, (\phi_{\bar{k}} \varepsilon^{\bar{k}\bar{m}\bar{n}})'\} (\phi_{\bar{r}} \phi_{\bar{s}})' + \{\mathcal{H}_{ij}, (\phi_{\bar{r}} \phi_{\bar{s}})'\} (\phi_{\bar{k}} \varepsilon^{\bar{k}\bar{m}\bar{n}})', \\
W_2 &= \{\mathcal{H}_{ij}, T'_{\perp \bar{m}\bar{n}}\} (\hat{\pi}^{(\bar{r}\bar{s})})' + \{\mathcal{H}_{ij}, (\hat{\pi}^{(\bar{r}\bar{s})})'\} T'_{\perp \bar{m}\bar{n}} + a_0 \phi \{B_{ij}, (\hat{\pi}^{(\bar{r}\bar{s})})'\} T'_{\perp \bar{m}\bar{n}}, \\
W_3 &= \{\mathcal{H}_{ij}, (\bar{V}^{\bar{r}})'\} (T^{\bar{s}}_{\bar{m}\bar{n}})' + \{\mathcal{H}_{ij}, (T^{\bar{s}}_{\bar{m}\bar{n}})'\} (\bar{V}^{\bar{r}})'. \tag{2.3}
\end{aligned}$$

Explicit results are obtained with the help of Appendix ??:

$$\begin{aligned}
W_1 &= \phi_{\bar{k}} (\delta_j^n \varepsilon_{\bar{i}}^{\bar{k}\bar{m}} - \delta_j^m \varepsilon_{\bar{i}}^{\bar{k}\bar{n}}) \delta \cdot \phi_{\bar{r}} \phi_{\bar{s}} + \phi_{\bar{i}} (g_{j\bar{r}} \phi_{\bar{s}} + g_{j\bar{s}} \phi_{\bar{r}}) \delta \cdot \phi_{\bar{k}} \varepsilon^{\bar{k}\bar{m}\bar{n}} - (i \leftrightarrow j), \\
W_2 &= W_{21} + W_{22} + W_{23}, \\
W_{21} &= \{\mathcal{H}_{ij}, T'_{\perp \bar{m}\bar{n}}\} (\hat{\pi}^{(\bar{r}\bar{s})})' = (n_i T_{j\bar{m}\bar{n}} + g_{jm} T_{\perp \bar{i}\bar{n}} - g_{jn} T_{\perp \bar{i}\bar{m}}) \hat{\pi}^{(\bar{r}\bar{s})} \delta \\
&\quad - (\vartheta_{j\gamma} \partial'_\beta \delta - \vartheta_{j\beta} \partial'_\gamma \delta) (n_i e_{\bar{m}}^\beta e_{\bar{n}}^\gamma \hat{\pi}^{(\bar{r}\bar{s})})' - (i \leftrightarrow j), \\
W_{22} &= \{\mathcal{H}_{ij}, (\hat{\pi}^{(\bar{r}\bar{s})})'\} T'_{\perp \bar{m}\bar{n}} = (\delta_j^{(r} \hat{\pi}_{\bar{i}}^{\bar{s})} + \delta_j^{(s} \hat{\pi}_{\bar{i}}^{\bar{r})}) T_{\perp \bar{m}\bar{n}} \delta - (i \leftrightarrow j), \\
W_{23} &= a_0 \phi \{B_{ij}, (\hat{\pi}^{(\bar{r}\bar{s})})'\} T'_{\perp \bar{m}\bar{n}} = a_0 \phi \partial_\alpha [(B_{ij}^{0\alpha} g^{r\bar{s}} + g^{\bar{r}k} B_{ki}^{0\alpha} \delta_j^{\bar{s}} + g^{\bar{r}k} B_{jk}^{0\alpha} \delta_i^{\bar{s}})] \\
W_3 &= W_{31} + W_{32}, \\
W_{31} &= \{\mathcal{H}_{ij}, (\bar{V}^{\bar{r}})'\} (T^{\bar{s}}_{\bar{m}\bar{n}})' = (T_{j\bar{i}}^{\bar{r}} + \delta_j^r \bar{V}_{\bar{i}}) T^{\bar{s}}_{\bar{m}\bar{n}} \delta - n_i T_{\perp \bar{j}}^{\bar{r}} (T^{\bar{s}}_{\bar{m}\bar{n}}) \delta \\
&\quad - (\vartheta_{j\gamma} \partial'_\beta \delta - \vartheta_{j\beta} \partial'_\gamma \delta) (e_{\bar{i}}^\beta e^{\bar{r}\gamma} T^{\bar{s}}_{\bar{m}\bar{n}})' - (i \leftrightarrow j), \\
W_{32} &= \{\mathcal{H}_{ij}, (T^{\bar{s}}_{\bar{m}\bar{n}})'\} (\bar{V}^{\bar{r}})' = [(g_{jm} T^{\bar{s}}_{\bar{i}\bar{n}} - g_{jn} T^{\bar{s}}_{\bar{i}\bar{m}}) - n_i (\delta_j^p n^s + \delta_j^s n^p) T_p \\
&\quad - \delta_{\bar{i}}^{\bar{s}} (\vartheta_{j\gamma} \partial'_\beta \delta - \vartheta_{j\beta} \partial'_\gamma \delta) (e_{\bar{m}}^\beta e_{\bar{n}}^\gamma \bar{V}^{\bar{r}})]' - (i \leftrightarrow j).
\end{aligned}$$

Step 2. The PB that we are looking for,

$$\{C_{ij}, \chi'\} = \underbrace{a_0(b\phi^{-1})'W_1(w'_2 + 2a_0J'\phi'w'_3)}_{Z_1} + \underbrace{a_0(b\phi^{-1})'w'_1(W_2 + 2a_0J'\phi'W_3)}_{Z_2=Z_{21}+Z_{22}},$$

can be calculated directly from (2.4). The term W_1 is proportional to the δ function, whereas W_2 and W_3 contain both δ and $\partial\delta$. Terms with $\partial\delta$ can be transformed using the δ -function identity (??)₁. The first term in (2.5) is given by

$$Z_{1ij} = 2a_0(b\phi^{-1})\phi_{\bar{k}} \left[\delta_j^n \varepsilon_i^{\bar{k}\bar{m}} \phi_{\bar{r}} \phi_{\bar{s}} + \phi_{\bar{i}} g_j(\bar{r}\phi_{\bar{s}}) \varepsilon^{\bar{k}\bar{m}\bar{n}} \right] \left(T_{\perp\bar{m}\bar{n}} \hat{\pi}^{(\bar{r}\bar{s})} + 2a_0 J \phi V^{\bar{r}} T^{\bar{s}}_{\bar{m}\bar{n}} \right) \delta - (i \leftrightarrow j)$$

The structure of the 2nd term is more complicated, with both δ and $\partial\delta$ terms. The contributions to $Z_2(\partial\delta)$ are found by isolating $\partial\delta$ terms in W_2 and W_3 :

$$W_2(\partial\delta) = - \left[(\vartheta_{j\gamma} \partial'_\beta \delta - \vartheta_{j\beta} \partial'_\gamma \delta) (n_i e_{\bar{m}}^\beta e_{\bar{n}}^\gamma \hat{\pi}^{(\bar{r}\bar{s})})' - (i \leftrightarrow j) \right] \\ + a_0 \phi \left[(B_{ij}^{0\alpha} g^{r\bar{s}} + g^{\bar{r}k} B_{ki}^{0\alpha} \delta_j^{\bar{s}} + g^{\bar{r}k} B_{jk}^{0\alpha} \delta_i^{\bar{s}}) \partial_\alpha \delta \right] T'_{\perp\bar{m}\bar{n}},$$

$$W_3(\partial\delta) = - (\vartheta_{j\gamma} \partial'_\beta \delta - \vartheta_{j\beta} \partial'_\gamma \delta) \left(e_i^\beta e^{\bar{r}\gamma} T^{\bar{s}}_{\bar{m}\bar{n}} + \delta_i^{\bar{s}} e_{\bar{m}}^\beta e_{\bar{n}}^\gamma \bar{V}^{\bar{r}} \right)' - (i \leftrightarrow j)$$

Now, one can insert these terms in (2.5), substitute the resulting expression $Z_2(\partial\delta; x, x')$ into Eq. (0.7) for the determinant, rearrange the result with the help of the δ -function identity (??)₁ and integrate over d^3x (applying the partial integration where needed). Then, replacing x' by x , one obtains the first term in the differential equation (0.8). The second term in (0.8) is produced by the δ function contributions from both Z_1 and Z_2 .