Covariance in Teleparallel Theories and Definitions of Energy-Momentum

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Covariant and Non-Covariant Formulation

Covariant formulation

Teleparallel theories where we use the tetrad and the purely inertial spin connection together and the theory is covariant with respect to both diffeomorphisms and local Lorentz transformations. The spin connection needs to be determined accordingly to the tetrad.

Non-Covariant formulation ("pure tetrad", Weitzenböck gauge,...)

Teleparallel theories where the spin connection is set to zero for all tetrads and we loose the local Lorentz covariance. There is a special class of proper/good tetrads that solve certain problems and need to be determined.

Both formulations are actually two pictures of the very same theory and are extremely closely related since finding the special class of tetrads is practically equivalent to finding the spin connection. Differences are very subtle but important.

Riemann Connection in Holonomic/Anholonomic Basis

Levi-Civita Linear Connection (Holonomic) a.k.a Christoffel symbols

$$\overset{\circ}{\Gamma}{}^{\rho}{}_{\nu\mu} = \frac{1}{2} g^{\rho\sigma} \left(g_{\nu\sigma,\mu} + g_{\mu\sigma,\nu} - g_{\nu\mu,\sigma} \right)$$

Levi-Civita Spin Connection (Anholonomic)

$$\overset{\circ}{\omega}{}^{a}{}_{b\mu} = \frac{1}{2} h^{\ c}{}_{\mu} \left[f_b{}^a{}_c + f_c{}^a{}_b - f^a{}_{bc} \right]$$

where we have the coefficients of anholonomy

$$f^c{}_{ab} = h_a{}^{\mu} h_b{}^{\nu} (\partial_\nu h^c{}_\mu - \partial_\mu h^c{}_\nu)$$

Both are related through

$$\overset{\circ}{\Gamma}{}^{\rho}{}_{\nu\mu} = h_a{}^{\rho} \partial_\mu h^a{}_\nu + h_a{}^{\rho} \overset{\circ}{\omega}{}^{a}{}_{b\mu} h^b{}_\nu$$

Teleparallel Geometry and Connection

A complimentary approach to Riemann/Levi Civita connection, where we define teleparallel connection ${\omega^a}_{b\mu}$ by:

1. **Zero Curvature (Teleparallel Condition)** implying that the connection takes the "pure-gauge" form

$$ {\omega^a}_{b\mu} = {\Lambda^a}_c \partial_\mu {(\Lambda^{-1})^c}_b $$

   where ${\Lambda^a}_c \subset GL(4)$

2. **Metric compatibility condition** implying that $\omega_{ab\mu} = -\omega_{ba\mu}$ from where follows that ${\Lambda^a}_b \subset SO(3,1))$

Teleparallel Spin Connection

$$ {\omega^a}_{b\mu} = {\Lambda^a}_c \partial_\mu {(\Lambda^{-1})^c}_b \qquad {\Lambda^a}_b \subset SO(3,1)) $$

Teleparallel Spin Connection and Inertia

Purely inertial connection

$$\omega^a{}_{b\mu} = \Lambda^a{}_c \partial_\mu (\Lambda^{-1})^c{}_b \qquad \Lambda^a{}_b \subset SO(3,1))$$

- Means that this connection represents only inertial effects, i.e. acceleration of frames
- Equivalent to the most general R/LC connection in Minkowski spacetime
- Most general tetrad in Minkowski spacetime is

  $$h_{ref}{}^a{}_\mu = \Lambda^a{}_b e^b{}_\mu$$

  where $e^a{}_\mu = \mathrm{diag}(1,1,1,1)$, is the Cartesian diagonal tetrad
- So the teleparallel connection can be always calculated as a R/LC connection of the reference tetrad

  $$\omega^a{}_{b\mu} = \overset{\circ}{\omega}{}^a{}_{b\mu}(h_{ref})$$

Pure-Gauge Connection and Non-Covariant Formulation

Pure-gauge form $\omega^a{}_{b\mu} = \Lambda^a{}_c \partial_\mu (\Lambda^{-1})^c{}_b$ implies that it **can** be transformed to zero

Weitzenböck Gauge

- Class of tetrads for which we can choose spin connection to vanish

$$\omega^a{}_{b\mu} = 0, \qquad \Gamma^\rho{}_{\nu\mu} = h_a{}^\rho \partial_\mu h^a{}_\nu + 0$$

- If tetrad and spin connection would be completely independent then we could do this for all tetrads
- Turns out that tetrad and spin connection are not independent because we have freedom to do LLT on the tetrad and only "similar" LLT on the spin connection
- If they are not completely independent, then turns out that transforming the spin connection to zero gives us preferred class of tetrads (proper/good).

Drawbacks of Non-Covariant Formulation

Mathematical Consistency: without $\omega^a{}_{b\mu}$ the "torsion tensor" is not a tensor, it is a coefficient of anholonomy

- Ricci theorem $\omega^a{}_{b\mu} - \overset{\circ}{\omega}{}^a{}_{b\mu} = K^a{}_{b\mu}$ says that difference of two connections is a tensor
- Without the teleparallel spin connection the contortion tensor $K^a{}_{b\mu} = -\overset{\circ}{\omega}{}^a{}_{b\mu}$ is a connection ?!?!

Preferred Proper/Good Tetrads: there is a special class of tetrads for which we obtain finite conserved charges, action (next slide)

- In TEGR this could be understandable as we do not expect to have everything invariant (too much covariance is bad for you, this talk later)
- In modified case, $f(T)$ gravity, becomes a problem: there are special class of tetrads that are solutions and other are not solutions (good and bad tetrads Tamanini&Bohmer 2012)
  - Problem appears in Minkowski case already

    $$e^a{}_\mu = \mathrm{diag}(1,1,1,1), \qquad \bar{e}^a{}_\mu = \mathrm{diag}(1,1,r,r\sin\theta),$$

    $e^a{}_\mu$ solves trivially $f(T)$ FE while $\bar{e}^a{}_\mu$ does not!
  - It was suggested that loss of LLI is some model of QG
  - But, how come it appears for all $f(T)$ models in Minkowski for all energies !?!? This is a problem of consistency, not a model of QG!

Effect of LL dofs in TEGR

- In TEGR the field equations are functions of the metric tensor (do not determine LL dofs of tetrad or purely inertial connection)

$$\partial_\sigma \left(h S_a{}^{\rho\sigma} \right) - \kappa \, h j_a{}^\rho , = h \overset{\circ}{G}_a{}^\rho$$

  where $\overset{\circ}{G}_a{}^\rho$ is Einstein (R/LC) tensor
- There are other quantities that do feel LL dofs

  Conserved Charges: total energy-momentum depend on the choice of the tetrad in non-covariant formulation or spin connection in covariant formulation
  Teleparallel Action: same goes for the teleparallel action that transforms us a total derivative with respect to LLT
- We can use these properties to fix the local Lorentz degrees of freedom in TEGR in both formulations

  Covariant: Find to each tetrad a corresponding spin connection
  Non-Covariant: Find a special class of tetrads

History of LL dofs in TG

Einstein 1928: 6 extra components of the tetrad were supposed to represent EM Faraday tensor

Møller 1961: when deriving energy-momentum definition for gravity, realized that TG describes gravity only; imposes 6 "supplementary conditions"

Kopczynski 1981: discuss purely inertial connections for TG(NGR); within MAG

Maluf 1995: 1st (according to my knowledge) calculation of energy of BH and 1st appearance of the "proper tetrad" (or triad in Hamiltonian formalism)

Obukhov & Pereira 2002: metric-affine approach and 1st time appears the spin connection for a BH

Obukhov, Rubilar, et. al. 2006-2008: Series of papers about Noether charges where the spin connection is used and motivated as an asymptotic limit of R/LC connection

Ferraro, Fiorini, et. al. 2006-2008: $f(T)$ gravity and the problem of determining LL dofs become a question of dynamics and crucial for $f(T)$ gravity

Determining Proper Tetrad and/or Spin Connection

Use the effects of LL dofs to fix them

Finite Conserved Charges Principle

Find a such combination of the tetrad and spin connection (covariant), or a special class of tetrads (non-covariant), that leaves us with finite conserved charges

Finite Action Principle

- Or analogously, that lead to a finite action
- Very similar (almost same) to previous approach but with few differences (later this talk)

- Practical approach was first introduced by Obukhov *et. al.* where we take the teleparallel spin connection reference tetrad to be an asymptotic limit of the LC connection. For asymptotically flat spacetimes we obtain a flat connection
- We can slightly generalize (but not too much) to the **switching-off gravity method** where we assume that there exists some reference (background) tetrad having the same inertial properties (works for FRWL) by "switching-off" gravity (**This is a "cookbook", nothing fundamental!!!**)

Energy-Momentum Problem for Gravity

In order to examine the approach of finite conserved charges, we need to understand the problem of energy-momentum in TEGR

Based on recent work:

1. Emtsova, E. D., Krššák, M.,Petrov, A. N., Toporensky, A. V.: *On Conserved Quantities for the Schwarzschild Black Hole in Teleparallel Gravity*, arXiv:2105.13312

2. See also my upcoming paper

Method 1: Energy-Momentum from Field Equations

- The field equations

$$\partial_\sigma \left(hS_a{}^{\rho\sigma}\right) - \kappa\, hj_a{}^\rho = h\kappa\Theta_a{}^\rho$$

- Defines naturally conservation law

$$\partial_\rho(hj_a{}^\rho + h\Theta_a{}^\rho) = 0$$

- that can be naturally integrated to obtain conserved quantities

$$P_a = -\int_\Sigma d^3x(hj_a{}^0 + h\Theta_a{}^0)$$

- Using Stoke's theorem and field equations

$$P_a = -\lim_{r\to\infty}\frac{1}{\kappa}\int_{\partial\Sigma} d^2x\, hS_a{}^{01}$$

Some Properties of P_a

- Quasi-invariant under diffeomorphisms (tensor density)
- **Non-covariant formulation:** transforms non-tensorially with respect to LLT
- Even in the **covariant formulation**: transforms as a vector and depends on the asymptotic limit ${h^a}_\mu$ since

$$P_a \propto \int h {h_a}^\mu {S_\mu}^{01}$$

  and ${S_\mu}^{01}$ is invariant under LLT, while ${h_a}^\mu$ is not (this can cause a divergence and certain problems, see later)
- How to understand and why not to prefer

$$P_\mu = - \lim_{r \to \infty} \frac{1}{\kappa} \int_{\partial \Sigma} d^2x \, h {S_\mu}^{01}$$

  that can be obtained from writing field equations in a fully spacetime form?

Method 2: Noether Charges Obukhov& Rubilar 2006

- Consider instead a general Noether charge generated by the vector field ξ^μ

$$\mathcal{P}(\xi) = \lim_{r\to\infty} \frac{1}{\kappa} \int_{\partial\Sigma} d^2x\, h\, \xi^\sigma S_\sigma{}^{01}$$

- Defines a true tensor with respect to both diffeomorphisms and local Lorentz transformations
- Introduces a new vector field ξ generating a conserved Noether charge
- How to choose ξ and attach meaning to this Noether charge. We have four natural options:
  1. **Killing vector**
  2. **Tetrad** $h^a{}_\mu$ (some component)
  3. **Coordinate basis** ∂_μ
  4. ξ **as (quasi) independent field characterizing observer**
- what is the meaning of each choice? which is preferable?

Option 1: ξ as a Killing vector field

$$\mathcal{P}(\xi) = \mathcal{P}(\xi, h^a{}_\mu, \omega^a{}_{b\mu}) = \lim_{r\to\infty} \frac{1}{\kappa} \int_{\partial\Sigma} d^2x\, h\, \xi^\sigma S_\sigma{}^{01}(h^a{}_\mu, \omega^a{}_{b\mu})$$

- Makes a perfect sense in metric theories where we try to associate energy-momentum with the spacetime itself Chen& Nester 1999
- However, in tetrad theories as TEGR, the problem is that $S_\rho{}^{\mu\nu}(h^a{}_\mu, \omega^a{}_{b\mu})$ depends on the observer
- For different observers, we have different values of $S_\rho{}^{\mu\nu}$ but the same metric and the same Killing vectors
- Can lead to some weird results, see the free-falling example later!

Option 2: ξ as a tetrad field I

$$\mathcal{P}(\xi) = \lim_{r\to\infty} \frac{1}{\kappa} \int_{\partial\Sigma} d^2x\, h\, \xi^\sigma\, S_\sigma{}^{01}$$

- We can identify $\xi_{(a)} = h_a$ and have a set of Noether charges

$$\mathcal{P}(\xi_{(a)}) = \lim_{r\to\infty} \frac{1}{\kappa} \int_{\partial\Sigma} d^2x\, h\, \xi_{(a)}{}^\sigma\, S_\sigma{}^{01}$$

  and hence we have

$$\mathcal{P}(\xi_{(a)}) = P_a$$

- Explains the origin of P_a in the Noether approach: a conserved quantity generated by the tetrad itself

Option 2: ξ as a tetrad field II

Non-covariant Formulation: we can uniquely identify ξ with the tetrad h^a in $S_\rho{}^{\mu\nu}$ and have $P_a(h^a_\mu)$

Covariant Formulation: covariance allows us to treat $\xi = h_a{}^\mu$ and the tetrad $h'_a{}^\mu$ independently and we essentially have

$$P_a = P_a(h^a{}_\mu, h'^a{}_\mu, \omega^a{}_{b\mu}) = \lim_{r\to\infty} \frac{1}{\kappa} \int_{\partial\Sigma} d^2x\, h\, h_a{}^\sigma S_\sigma{}^{01}(h'^a{}_\mu, \omega^a{}_{b\mu})$$

where $h'^a = \Lambda^a{}_b h^b$ is some rotated tetrad

- Not only allows us to do this but prevents us to make a clear identification.
- While P_a is well-defined in non-covariant formulation, it is not-unique in the covariant formulation
- See the example later!

Option 3: ξ as a coordinate basis [wrong]

$$\mathcal{P}(\xi) = \lim_{r\to\infty} \frac{1}{\kappa} \int_{\partial\Sigma} d^2x\, h\, \xi^\sigma S_\sigma{}^{01}$$

- We can also try $\xi_{(a)}$ to identify with the coordinate basis e_a, i.e. $\xi_{(a)} = \delta^\mu_{(a)}$ and we obtain P_μ
- So P_μ is a Noether current generated by the vector fields $\xi_{(a)}$ identified with the coordinate basis
- Superpotential $S_\rho{}^{\mu\nu}$ depends still on the choice of the observer
- P_μ does not make much sense!

Option 4: ξ as independent field of observer

$$\mathcal{P}(\xi, h^a{}_\mu, \omega^a{}_{b\mu}) = \lim_{r\to\infty} \frac{1}{\kappa} \int_{\partial\Sigma} d^2x \, h \, \xi^\sigma S_\sigma{}^{01}(h^a{}_\mu, \omega^a{}_{b\mu})$$

- Treat ξ quasi-independent from $h^a{}_\mu$ but in a such way that ξ is a velocity of the observer representing the "physical nature" of the fields in $S_\sigma{}^{01}(h^a{}_\mu, \omega^a{}_{b\mu})$ (see examples that follows)
- Results in real scalar quantities in covariant formulation (but we pay the price of the above awkwardness)

Example: Schwarzschild BH (Non-Covariant) I

- Diagonal tetrad of the **static** observer

$$\overset{B}{h}{}^{a}{}_{\mu} \equiv \mathrm{diag}\Big(f^{\frac{1}{2}}, f^{-\frac{1}{2}}, r, r\sin\theta\Big) \qquad f = 1 - \frac{2M}{r}$$

  leads to divergent conserved charges
- Proper tetrad for a static observer

$$\overset{A}{h}{}^{a}{}_{\mu} \equiv \begin{bmatrix} f^{\frac{1}{2}} & 0 & 0 & 0 \\ 0 & f^{-\frac{1}{2}}\sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ 0 & f^{-\frac{1}{2}}\sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ 0 & f^{-\frac{1}{2}}\cos\theta & -r\sin\theta & 0 \end{bmatrix}$$

- And calculate the energy-momentum $P_a = (M, 0, 0, 0)$ (as well as P_μ)
- We can also consider Killing vector $\xi^\mu = (-1, 0, 0, 0)$ and show $\mathcal{P}(\xi) = M$

Example: Schwarzschild BH (Non-Covariant) II

- Consider a boost $\tilde{\Lambda}^a{}_b(\beta)$ with rapidity $\beta = \sqrt{\frac{2M}{r}}$ and define a **free-falling tetrad** $\overset{C}{h}{}^a{}_\mu = \tilde{\Lambda}^a{}_b(\beta)\overset{B}{h}{}^b{}_\mu$
- We can find **free-falling proper tetrad**:

$$\overset{D}{h}{}^a{}_\mu = \begin{pmatrix} 1 & \frac{\sqrt{2Mr}}{r-2M} & 0 & 0 \\ \sqrt{\frac{2M}{r}}\cos\varphi\sin\theta & \frac{r\cos\varphi\sin\theta}{r-2M} & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sqrt{\frac{2M}{r}}\sin\theta\sin\varphi & \frac{r\sin\theta\sin\varphi}{r-2M} & r\cos\theta\sin\varphi & r\cos\varphi\sin\theta \\ \sqrt{\frac{2M}{r}}\cos\theta & \frac{r\cos\theta}{r-2M} & -r\sin\theta & 0 \end{pmatrix}$$

- And consider the velocity of the free-falling observer

$$\tilde{\xi}^\mu = -\overset{D}{h}{}_{\hat{0}}{}^\mu = (-1/f, \sqrt{1-f}, 0, 0)$$

- We can calculate Noether charges and P_a
  1. ξ **(Killing vector)** leads to $\mathcal{P}(\xi) = 2M$ (meaning?)
  2. $\tilde{\xi}$ **(velocity of the observer)** leads to $\mathcal{P}(\xi) = 0$ and $P_a = (0,0,0,0)$ and hence make sense for the free-falling observer (equivalence principle)

Example: Schwarzschild BH (Covariant) I

- **Static observer** can be described either using the proper tetrad $\overset{A}{h}{}^{a}{}_{\mu}$ or the diagonal tetrad $\overset{B}{h}{}^{a}{}_{\mu}$ with the appropriate spin connection

$$\omega^{\hat{1}}{}_{\hat{2}2} = -\omega^{\hat{2}}{}_{\hat{1}2} = -1, \quad \omega^{\hat{1}}{}_{\hat{3}3} = -\omega^{\hat{3}}{}_{\hat{1}3} = -\sin\theta, \quad \omega^{\hat{2}}{}_{\hat{3}3} = -\omega^{\hat{3}}{}_{\hat{2}3} = -\cos\theta$$

  both lead to the same superpotential $S_{\rho}{}^{\mu\nu}$. Therefore, both $\{\overset{A}{h}{}^{a}{}_{\mu}, 0\}$ and $\{\overset{B}{h}{}^{a}{}_{\mu}, \omega^{a}{}_{b\mu}\}$ are physically equivalent and we call this the **static gauge**
- **Free-falling observer** can be described either using the free-falling proper tetrad $\overset{D}{h}{}^{a}{}_{\mu}$ or the diagonal tetrad $\overset{C}{h}{}^{a}{}_{\mu}$ with the same spin connection as in static case above. Therefore, both $\{\overset{D}{h}{}^{a}{}_{\mu}, 0\}$ and $\{\overset{C}{h}{}^{a}{}_{\mu}, \omega^{a}{}_{b\mu}\}$ are physically equivalent and we call this the **Lemaitre gauge**
- We have two choices of ξ as either
  1. $\xi^{\mu} = (-1, 0, 0, 0)$ **(Killing vector field and velocity of the static observer at infinity** $\xi^{\mu} = -\lim_{r\to\infty} \overset{A}{h}{}_{\hat{0}}{}^{\mu}$ **)**
  2. $\tilde{\xi}^{\mu} = -\overset{D}{h}{}_{\hat{0}}{}^{\mu} = (-1/f, \sqrt{1-f}, 0, 0)$ **velocity of a free-falling observer**

Example: Schwarzschild BH (Covariant)II

- Noether charges can be summarized as

Gauge	$\mathcal{P}(\xi)$	$\mathcal{P}(\tilde{\xi})$
Static	**M**	M
Lemaitre	$2M$	**0**

- Well-defined quantities:

  $\mathcal{P}(\xi)$ **in static gauge:** ξ is both Killing vector and velocity of the static observer at infinity. Meaningful energy for BH

  $\mathcal{P}(\tilde{\xi})$ **in Lemaitre (free-fall) gauge:** $\tilde{\xi}$ is a velocity of a free-falling observer and and hence is well-defined quantity for free-fall gauge

- Ill-defined quantities:

  $\mathcal{P}(\tilde{\xi})$ in static gauge: using velocity of free-fall observer to calculate energy of static observer? Ill-defined despite nice answer M

  $\mathcal{P}(\xi)$ in Lemaitre gauge: if we consider ξ as a velocity of a static observer at infinity, why to calculate

- **Problem of P_a in covariant formulation** is that it allows to consider ill-defined quantities like $\mathcal{P}(\tilde{\xi})$ in the static gauge
- In Lemaitre gauge, covariant charge P_a leads to non-unique results for other components

$$P_a(\overset{D}{h}{}^a{}_\mu, 0) = (0,0,0,0), \qquad P_a(\overset{C}{h}{}^a{}_\mu, \omega^a{}_{b\mu}) = (0,\infty,0,0)$$

Conclusions about Energy-Momentum in TG

- Noether approach allows in principle to choose ξ as either Killing vector field or a tetrad
- The case of free-falling observer and the result $2M$ disfavour the Killing vector choice
- For tetrad theories is more natural to consider the Noether charges generated by the tetrads themselves
- Non-covariant energy-momentum $P_a(h^a{}_\mu)$ uniquely identifies ξ with a tetrad field in the action
- Covariant version of P_a is problematic as it allows to have different tetrads in the same expression $P_a(h^a{}_\mu, h'^a{}_\mu, \omega^a{}_{b\mu}) = P_a(h^a{}_\mu, h^a{}_\mu, \omega'^a{}_{b\mu})$
- In the covariant formulation makes more sense to use the Noether charge $\mathcal{P}(\xi, h^a{}_\mu, \omega^a{}_{b\mu})$ and make sure that ξ represents the observer from the same gauge (static observer in static gauge, free-falling observer in free-fall gauge)

Problems of Finite Charges Approach

- Previous discussions shows that the discussion of finite charges is still quite complicated topic
- Beside requirement of finiteness of $\mathcal{P}$ (or P_a?) there is already requirement
- Difficult to use when we do not know the answer already. Example of $2M$ shows that some finite answers are probably wrong.
- In non-static and asymptotically non-flat situation the conserved charges do not have to exist at all
- Example: FRWL spacetime (is not asymptotically flat and hence does not define conserved charge automatically)

Finite Action Principle and LL dofs Krssak&Pereira 2015, Krssak 2016, ...

- Instead of finite conserved charges, consider the finiteness of the action as guiding principle

$$\lim_{r\to\infty} T(h^a{}_\mu, \omega^a{}_{b\mu}) = \mathcal{O}\left(\frac{1}{r^4}\right)$$

that results in a finite (Euclidean) action
- Does not depend on details of definition of conserved charges
- Hence unifies covariant and non-covariant approaches: "proper" tetrads are tetrads that lead to this finite action, or we have to use spin connection that bring action to being finite
- Since $\omega^a{}_{b\mu} = \overset{\circ}{\omega}{}^a{}_{b\mu}(h_{ref})$ where h_{ref} is a *reference* or *background* tetrad and

$$\mathcal{L}_{TG}(h^a{}_\mu, \omega^a{}_{b\mu}) = \mathcal{L}_{TG}(h^a{}_\mu, 0) + \frac{1}{\kappa}\partial_\mu\left(h\omega^{\rho\mu}{}_\rho\right)$$

is equivalent to adding a boundary term
- Analogue of *background subtraction* by Gibbons&Hawking 1974

$$\mathcal{S}_{ren}(g_{\mu\nu}, g^{ref}_{\mu\nu}) = \mathcal{S}_{EH}(g_{\mu\nu}) + \mathcal{S}_{GHY}(g_{\mu\nu}) - \mathcal{S}_{GHY}(g^{ref}_{\mu\nu})$$

Non-Uniqueness and Remnant Symmetries

How unique is determination of LL dofs based on the action?

- Not all local LLTs do enter the action

$$\mathcal{L}_{TG}(h^a{}_\mu, \omega^a{}_{b\mu}) = \mathcal{L}_{TG}(h^a{}_\mu, 0) + \frac{1}{\kappa}\partial_\mu \left(h\omega^{\rho\mu}{}_\rho\right)$$

- Only 4 components $\omega^{\rho\mu}{}_\rho$ do enter the action and we have 6 LLTs
- There aways do exist LLTs that leave the action invariant!

Remnant Symmetries Ferraro&Fiorini 2014

A class of local Lorentz transformations that do not change $\omega^{\rho\mu}{}_\rho$ and hence leave action invariant. (In non-covariant formulation: a class of LLTs that do not change the action)

- Remnant symmetries do not form a group
- Remnant symmetries are highly specific for each tetrad
- See my upcoming paper!

Consequences of Remnant Symmetries

- In non-covariant formulation imply that there the action is not given for an observer uniquely but rather for a class of observers related by remnant symmetries
- In covariant formulation imply that the spin connection is not determined uniquely for an observer but only up to a remnant symmetry
- Example: the static and free-falling tetrad lead to a same spin connection. Therefore

$$\overset{B}{h}{}^{a}{}_{\mu} \equiv \text{diag}\Big(f^{\frac{1}{2}}, f^{-\frac{1}{2}}, r, r\sin\theta\Big)$$

and

$$\overset{C}{h}{}^{a}{}_{\mu} = \tilde{\Lambda}^{a}{}_{b}(\beta)\overset{B}{h}{}^{b}{}_{\mu}$$

with the boost $\tilde{\Lambda}^{a}{}_{b}(\beta)$ with rapidity $\beta = \sqrt{\frac{2M}{r}}$ are related by a remnant symmetry (i.e. $\tilde{\Lambda}^{a}{}_{b}(\beta)$ is a remnant symmetry)
- Important consequences for $f(T)$ dynamics and nature of degrees of freedom!

Summary and Outlook

- Both covariant and non-covariant formulations are extremely closely related since finding the special class of tetrads is practically equivalent to finding the spin connection for the tetrad (My preference is for covariant formulation for the sake of mathematical consistency and using proper tensors)
- We need to understand how to fix LL dofs in both covariant and non-covariant formulation
- We have in principle two possibilities either finite conserved chargers or finite action (my preference)
- Crucial problem: How to implement them in practice? The **switching-off method** is definitely not fundamental and we need something better
- Crucial role played by remnant symmetries (non-uniqueness of the tetrad/ non-uniqueness of the spin connection)
- Detailed analysis show that despite both approaches being equivalent, there are subtleties with definition of energy-momentum that show some differences and we need to treat both cases separately (P_a in non-covariant and $\mathcal{P}(\xi)$ in covariant)
- Further research needed!