Gauge transformations and Lorentz invariance
A geometric view on teleparallel gravity

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Why use tetrads in observations?

- Every observer can establish local frame of reference at \( x \in M \):
  - Four-velocity of observer \( \rightsquigarrow \) direction of time component.
  - Clock showing proper time \( \rightsquigarrow \) normalization of time component.
  - Light rays / radar experiment \( \rightsquigarrow \) direction of spatial components.
  - Light turnaround time \( \rightsquigarrow \) normalization of spatial components.
  - Parity-violating particles \( \rightsquigarrow \) orientation of frame.

\( \Rightarrow \) Established frames are related by proper Lorentz transformation.

\( \Rightarrow \) Frames can be chosen independently at every point \( x \in M \).

\( \Rightarrow \) "Fields of frames" related by local Lorentz transformation.

\( \Rightarrow \) Measurements of frequency, distance, time etc relative to frame.

\( \Rightarrow \) Observed quantities, in general, depend on choice of frame.

\( \Rightarrow \) Need prescription to translate quantities between different frames.
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Inertial observers and frames

- Observers follow world lines $\gamma : \mathbb{R} \rightarrow M$. 

\[ \begin{align*}
\text{Inertial observers: tangent } \dot{\gamma}^\mu &= e_0^\mu \text{ follows parallel transport:} \\
&= d e_0^\mu d\tau + \nabla^\mu \Gamma_{\nu\rho} e_0^\nu e_0^\rho = 0 \\
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\end{align*} \]

$\Rightarrow$ Inertial frame defined along world line. 

- World lines of initially separated inertial observers may cross. 
- Inertial observer world lines may not form congruences. 

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  \]
- Inertial frame: frame $e_a^{\mu}$ follows parallel transport:
  \[
  \frac{de_a^{\mu}}{d\tau} + \tilde{\Gamma}^{\mu}_{\nu\rho} e_a^{\nu} e_0^{\rho} = 0. \tag{2}
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What is local Lorentz covariance?

Statement of local Lorentz covariance

Observable, physical quantities are Lorentz covariant, i.e., at every point \( x \in M \) of spacetime \( M \) the physical quantities \( Q, Q' \) measured at \( x \) with respect to orthonormal frames \( \theta, \theta' \), which are related to each other by a (proper) Lorentz transformation \( \Lambda \in SO_0(1, 3) \), \( \theta = \Lambda \theta' \), are related to each other by some representation \( \rho : SO_0(1, 3) \rightarrow GL(n) \) of the Lorentz group, \( Q = \rho(\Lambda)Q' \).
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Consequence of local Lorentz covariance

Observable, physical fields are described by sections of bundles associated to the orthonormal frame bundle via their corresponding representation \( \rho \), i.e., they are tensor fields.
Common lore: *One can always use the Weitzenböck gauge.*

- The spin connection is flat:
  \[
  \partial_\mu \omega^a_{b\nu} - \partial_\nu \omega^a_{b\mu} + \omega^a_{c\mu} \omega^c_{b\nu} - \omega^a_{c\nu} \omega^c_{b\mu} \equiv 0. \tag{3}
  \]

  ⇒ *The spin connection can always be written in the form*

  \[
  \omega^a_{b\mu} = \Lambda^a_c \partial_\mu (\Lambda^{-1})^c_b. \tag{4}
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  ⇒ *One can achieve the Weitzenböck gauge by*

  \[
  \theta^a_\mu = \Lambda^a_b \theta^b_\mu. \]

- **Questions posed by the adept of geometry:**
  1. How can we determine the transformation \(\Lambda_{ab}\)?
  2. Is this even true?

- **Remark:** this holds also in symmetric and general teleparallelism.
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- \(\Lambda^{a}_{b}\) and \(\bar{w}^{a}_{\mu}\) defined only up to global transform

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  \Lambda^{a}_{b} \mapsto \Lambda'^{a}_{b} = \Lambda^{a}_{c} \Omega^{c}_{b}, \quad \bar{w}^{a}_{\mu} \mapsto \bar{w}'^{a}_{\mu} = (\Omega^{-1})^{a}_{b} \bar{w}^{b}_{\mu}. \tag{5}
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How to obtain the Weitzenböck gauge?

• Recall that we have gauge invariant quantities:
  ○ The metric $g_{\mu\nu} = \eta_{ab} \theta^a_\mu \theta^b_\nu$.
  ○ The teleparallel affine connection $\Gamma^\mu_{\nu\rho} = e^a_\mu (\partial_\rho \theta^a_\nu + \omega^a_{b\rho} \theta^b_\nu)$.
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- The tetrad and connection satisfy the “tetrad postulate”:
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  \partial_\mu \theta^a_{\nu} + \omega^a_{b\mu} \theta^b_{\nu} - \Gamma^\rho_{\nu\mu} \theta^a_{\rho} = 0. \quad (6)
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- Recipe for integrating the connection:
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⇒ Recipe for integrating the connection:
  1. Choose \( \theta^a_{\mu}(x) \) at some \( x \in M \) to fit with the metric.
  2. For any other \( y \in M \), choose path \( x \leadsto y \), and parallel transport.
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- Obtained tetrad satisfies required properties:
  $\checkmark$ $\theta^a_\mu$ gives correct metric, since connection is metric-compatible.
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- Obtained tetrad satisfies required properties:
  \( \checkmark \) \( \theta^a_\mu \) gives correct metric, since connection is metric-compatible.
  \( \checkmark \) Global Lorentz invariance encoded in freedom of choice for \( \theta^a_\mu (x) \).
Can we always use the Weitzenböck gauge?

- Recipe for integrating the connection:
  1. At some $x \in M$,
Can we always use the Weitzenböck gauge?

- Recipe for integrating the connection:
  1. At some \( x \in M \), choose \( \bar{\theta}^a_\mu (x) \) to fit with the metric.

\[
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Recipe for integrating the connection:

1. At some \( x \in M \), choose \( \omega^a_\mu(x) \) to fit with the metric.
2. For any other \( y \in M \), choose path \( x \xrightarrow{\gamma} y \),

\[
M R_{\mu \nu \rho \sigma} = 0
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- Recipe for integrating the connection:
  1. At some $x \in M$, choose $\tilde{w}^a_{\mu}(x)$ to fit with the metric.
  2. For any other $y \in M$, choose path $x \xrightarrow{\gamma} y$, and parallel transport.

$M_{\mu\nu\rho\sigma} = 0$

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- **What happens if we choose another path \( x \xrightarrow{\gamma'} y \)?**
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  ✓ Vanishing curvature: parallel transport along both path agrees.

$$R^\mu_{\nu\rho\sigma} = 0$$
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- What happens if we choose another path $x \xrightarrow{\gamma'} y$?
  ✓ Vanishing curvature: parallel transport along both path agrees.
  ☞ But only if $\gamma$ and $\gamma'$ are homotopic paths!
Trouble with the tetrad?

- Starting from an arbitrary tetrad and flat spin connection:
  - One may always locally transform into Weitzenböck gauge.
  - One may not always globally transform into Weitzenböck gauge.
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- Starting from an **arbitrary tetrad and flat spin connection**:
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- Is there always some global tetrad and flat spin connection?
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  - We want to be able to describe spinor fields on spacetime.
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- The case of the tetrad:
  - We want to be able to describe spinor fields on spacetime.
  - Physical spacetime manifold must admit a spin structure.
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Consider local Lorentz transformations $\Lambda : M \rightarrow \text{SO}(1,3)$:

- Simultaneous action on tetrad and spin connection:
  
  $$ (\theta, \omega) \mapsto (\Lambda \theta, \Lambda \omega \Lambda^{-1} + \Lambda d\Lambda^{-1}). $$

- $(\theta, \omega) \sim (\theta', \omega')$ if and only if $(g, \Gamma) = (g', \Gamma')$.

$\Rightarrow$ Orbits parametrized by metric and teleparallel affine connection.

- Decomposition of the Lorentz group:
  - Proper Lorentz group $\text{SO}_0(1,3) \subset \text{SO}(1,3)$, $T, P \in \text{SO}(1,3)$.
  - Standard model of particle physics only invariant under $\text{SO}_0(1,3)$.

$\Rightarrow$ Need orientation and time orientation in addition to $g$ and $\Gamma$.

$\Rightarrow$ Physical geometries parametrized by orbits of $\text{SO}_0(1,3)$. 

- Physical geometry: $\text{SO}_0(1,3)$ reduction of the frame bundle & $\Gamma$. 

Manuel Hohmann (University of Tartu)
Palatini and the space of orbits

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- Consider locally $\text{SO}(1, 3)$-invariant teleparallel gravity theory:
  - $\Lambda : M \rightarrow \text{SO}(1, 3)$ maps solutions to solutions.
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What about the teleparallel affine connection?

- Coupling of the teleparallel affine connection $\Gamma$:
  - No direct coupling with matter (commonly considered consistent).
  - Possible coupling to metric through gravity (vanishes in TEGR).

$\Rightarrow$ Teleparallel connection becomes just (another) "dark" field:
- Scalar fields / dark energy in scalar-tensor theories.
- "Dark" vector fields, "dark" photons in generalized Proca theories.
- Second metric in bimetric theories.

$\Rightarrow$ The "usual rules" for playing with "dark" fields apply:
- Find out which degrees of freedom couple to physical observables.
- "Remnant symmetries" may yield gauge degrees of freedom.
- Make sure physical degrees of freedom obey healthy evolution.

_PWM_ Pay attention to possible pathologies:
- Is the evolution of physical degrees of freedom determined?
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Dynamical field variables in teleparallel gravity

- What are the dynamical field variables in teleparallel gravity?
  1. Only a tetrad.

Problems encountered with choice of variables:
- Does not reflect observed local Lorentz invariance.
- Contains unphysical gauge degrees of freedom as variables.
- Does not contain information on orientation and time orientation.

Can we still use any of the other field variables?
- If (time) orientation is fixed, metric and connection are sufficient.
- Possible to choose tetrad and spin connection as representatives.
- Locally possible to transform into Weitzenböck gauge.

⇒ Most fundamental variables found in geometric picture.
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⇒ Most fundamental variables found in geometric picture.
1. Start with the general linear frame bundle $\pi : \text{GL}(M) \to M$. 

![Diagram of GL(M)]
The geometric picture

1. Start with the general linear frame bundle \( \pi : \text{GL}(M) \to M \).
2. Metric reduces bundle to orthonormal frame bundle \( \tilde{P} \).

\[
\text{GL}(M) \xrightarrow{\pi} M \quad \text{and} \quad \tilde{P}
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Lorentz invariance and geometry
TeleWorSe 2021
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4. Connection specifies horizontal directions \( TP = VP \oplus HP \) in \( \mathcal{P} \).
Conclusion

1. Physical observations single out frames which are:
   - Orthonormal - by using clocks, measuring rods, simultaneity.
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2. Physically observable geometry to be determined by gravity.

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Mantra
In order to understand gravity, one must understand geometry.

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Extra: the associated bundle

\[ P_x \times F \]

\[ \{p\} \times F \]

\[ (p, f) \mapsto p \]

\[ (p, f) \mapsto [p, f] \]

\[ (P \times_F F)_x \]
Extra: the many faces of connections

\[ j_{\pi(e)}^1 \sigma = \omega(e) \]

\[ \theta(w) = w_V \]

\[ \theta \]

\[ w \]

\[ \eta(e, v) \]

\[ \eta(e, \cdot) \]

\[ w_H \]

\[ \sigma_* \]

\[ \sigma_* (v') \]