Gauge transformations and Lorentz invariance

A geometric view on teleparallel gravity

Manuel Hohmann

Laboratory of Theoretical Physics - Institute of Physics - University of Tartu
Center of Excellence “The Dark Side of the Universe”

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Why use tetrads in observations?

- Every observer can establish local frame of reference at $x \in M$:
  - Four-velocity of observer $\rightsquigarrow$ direction of time component.
  - Clock showing proper time $\rightsquigarrow$ normalization of time component.
  - Light rays / radar experiment $\rightsquigarrow$ direction of spatial components.
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$\Rightarrow$ Need prescription to translate quantities between different frames.

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$\lightning$ Inertial frames in general not extendable beyond world line.

What is local Lorentz covariance?

Statement of local Lorentz covariance

Observable, physical quantities are Lorentz covariant, i.e., at every point $x \in M$ of spacetime M the physical quantities Q, Q' measured at x with respect to orthonormal frames θ, θ', which are related to each other by a (proper) Lorentz transformation $\Lambda \in \mathrm{SO}_0(1,3)$, $\theta = \Lambda\theta'$, are related to each other by some representation $\rho : \mathrm{SO}_0(1,3) \to \mathrm{GL}(n)$ of the Lorentz group, $Q = \rho(\Lambda)Q'$.

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Consequence of local Lorentz covariance

Observable, physical fields are described by sections of bundles associated to the orthonormal frame bundle via their corresponding representation ρ, i.e., they are tensor fields.

The Weitzenböck gauge

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  - The spin connection is flat:

$$\partial_\mu \omega^a{}_{b\nu} - \partial_\nu \omega^a{}_{b\mu} + \omega^a{}_{c\mu}\omega^c{}_{b\nu} - \omega^a{}_{c\nu}\omega^c{}_{b\mu} \equiv 0\,. \tag{3}$$

$\Rightarrow$ *The spin connection can always be written in the form*

$$\omega^a{}_{b\mu} = \Lambda^a{}_c \partial_\mu (\Lambda^{-1})^c{}_b\,. \tag{4}$$

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$$\Lambda^a{}_b \mapsto \Lambda'^a{}_b = \Lambda^a{}_c \Omega^c{}_b\,, \quad \overset{w}{\theta}{}^a{}_\mu \mapsto \overset{w}{\theta}{}'^a{}_\mu = (\Omega^{-1})^a{}_b \overset{w}{\theta}{}^b{}_\mu\,. \tag{5}$$

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- Remark: this holds also in symmetric and general teleparallelism.

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- Recall that we have gauge invariant quantities:
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- Obtained tetrad satisfies required properties:
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  - ✓ Global Lorentz invariance encoded in freedom of choice for $\overset{w}{\theta}{}^a{}_\mu(x)$.

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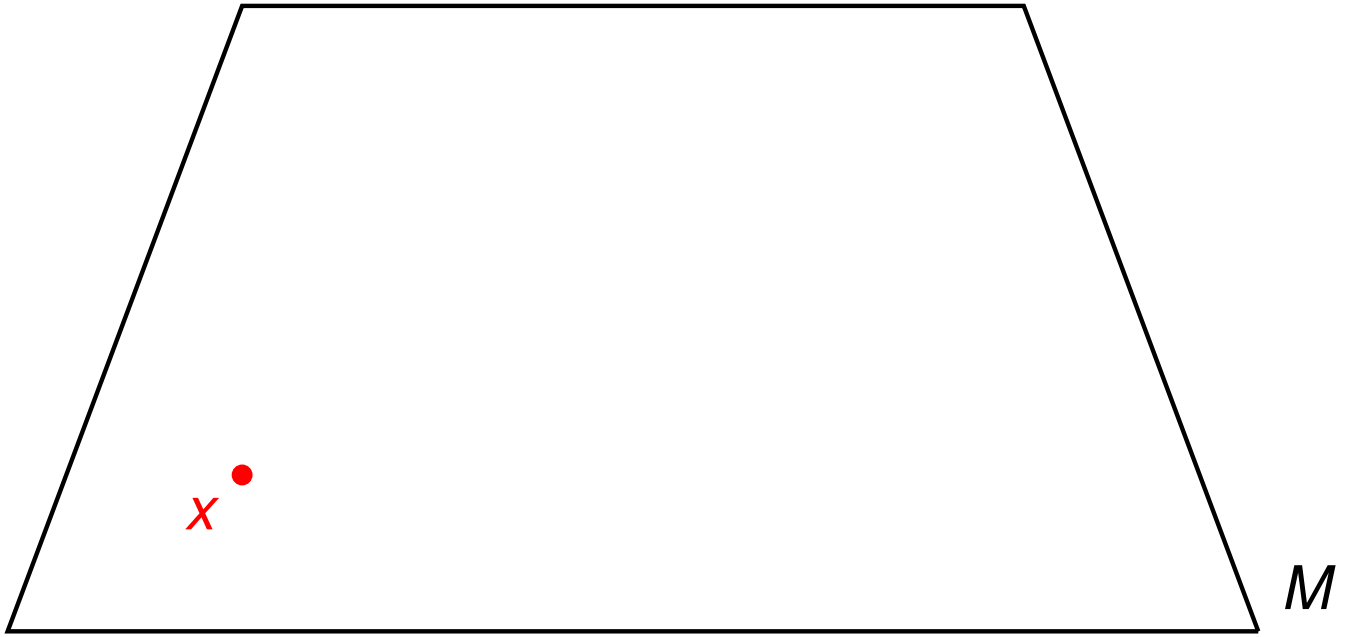

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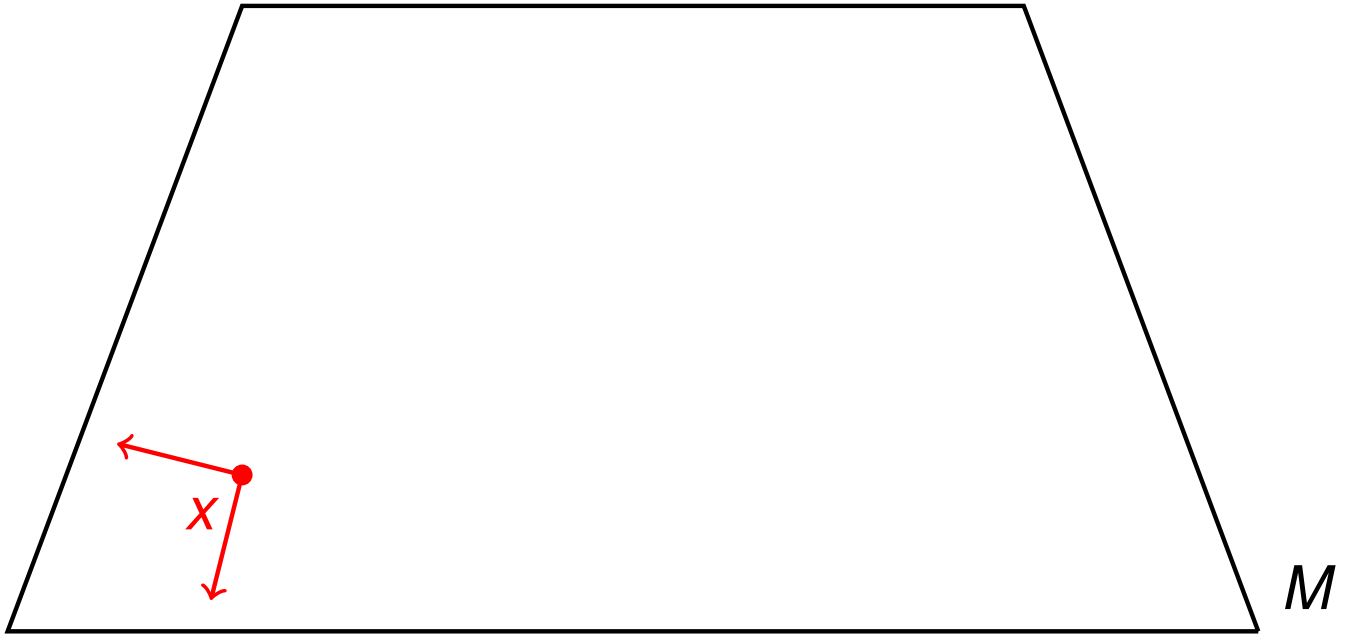

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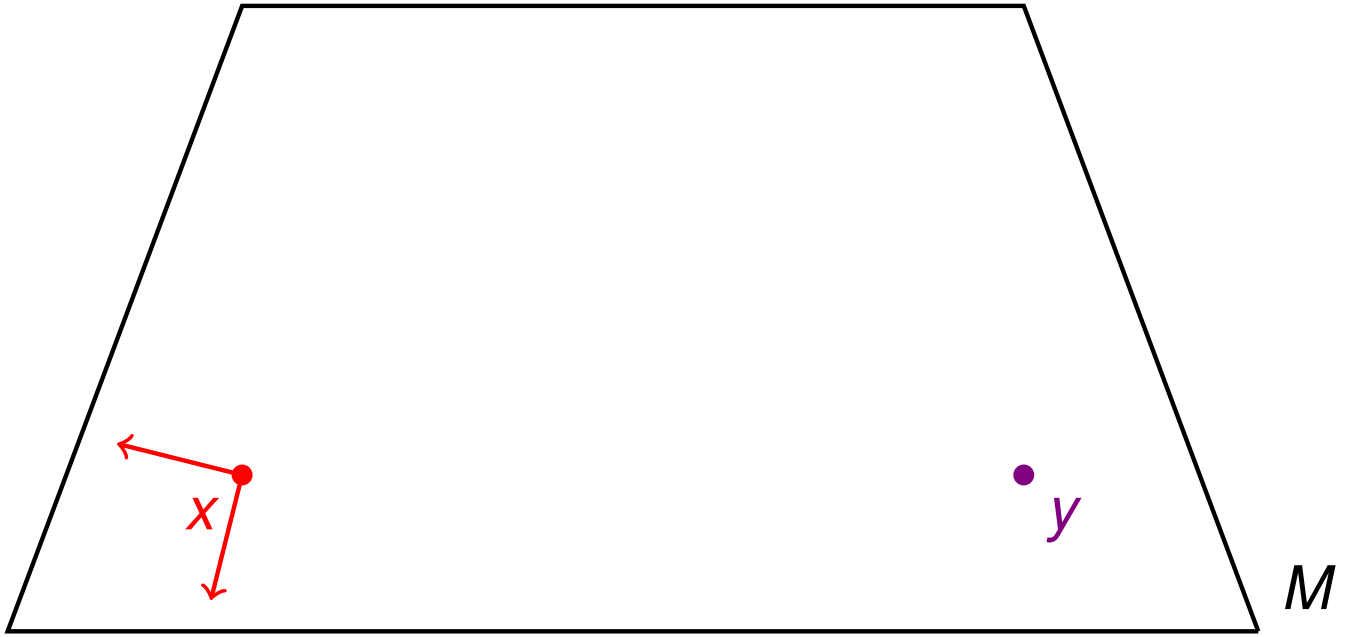

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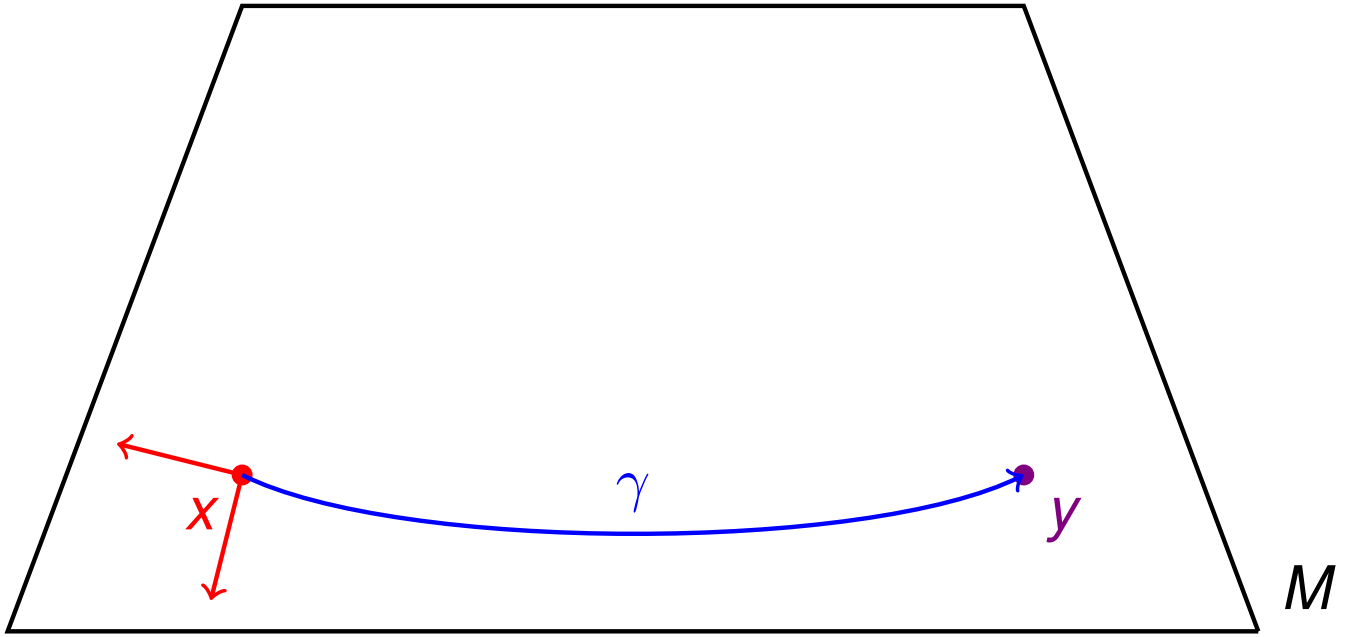

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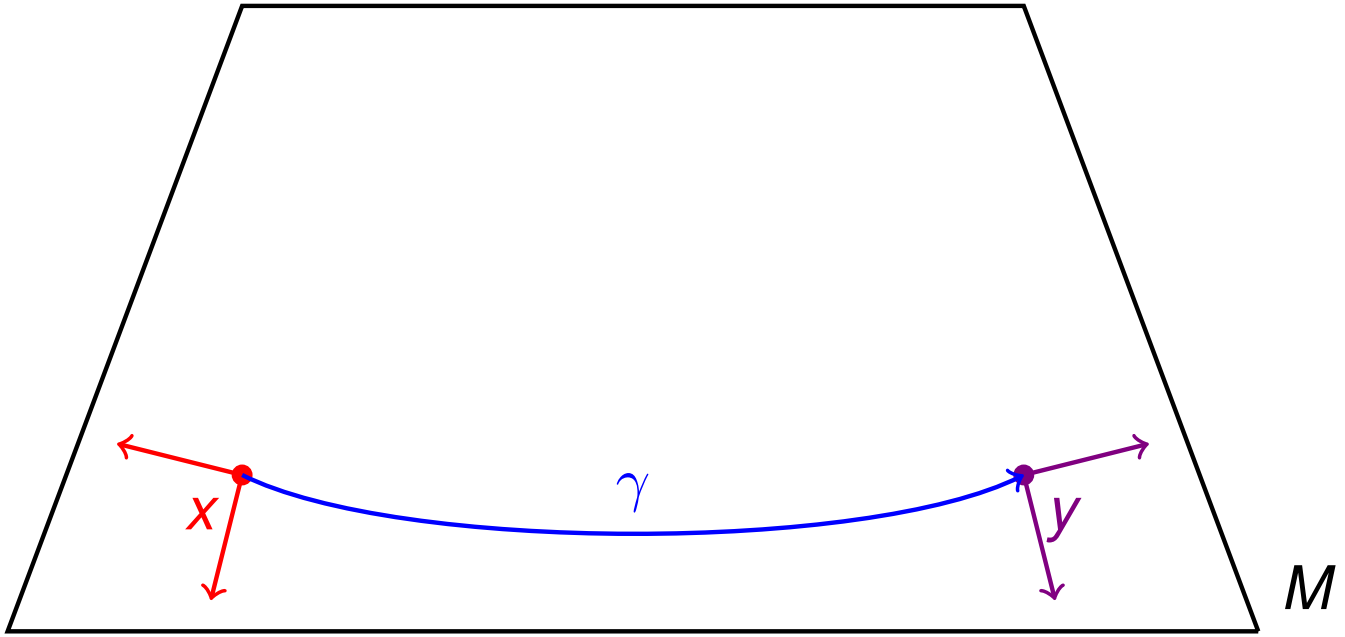

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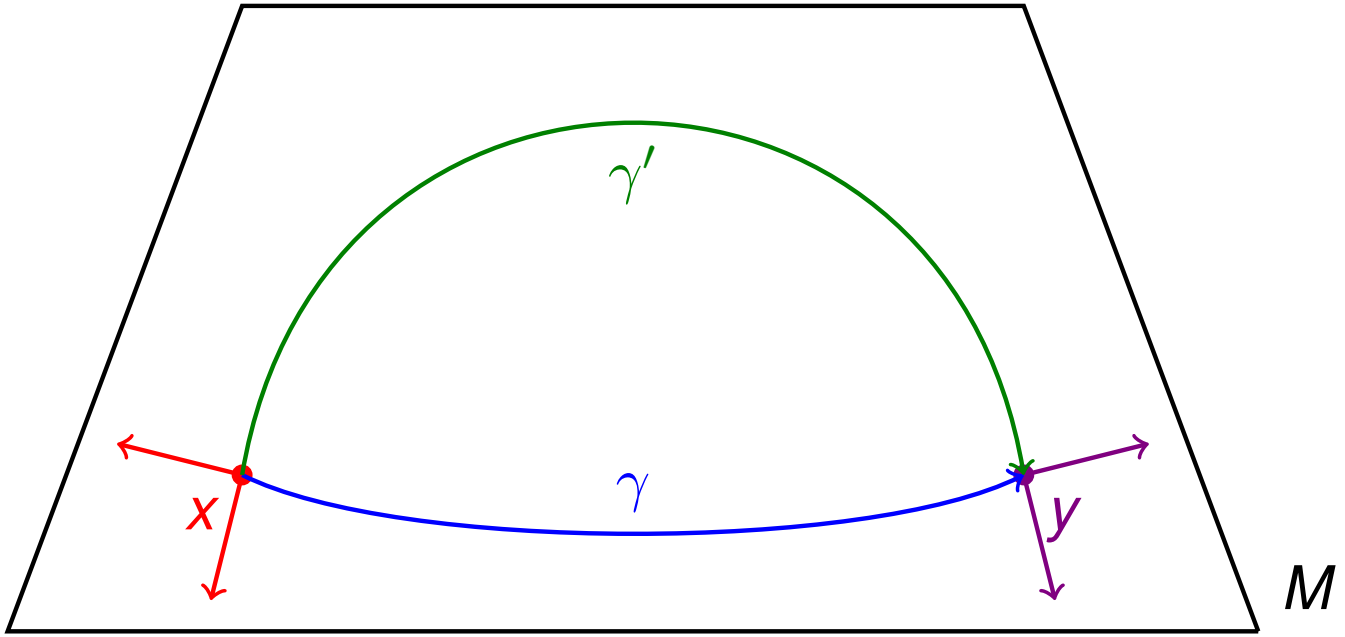

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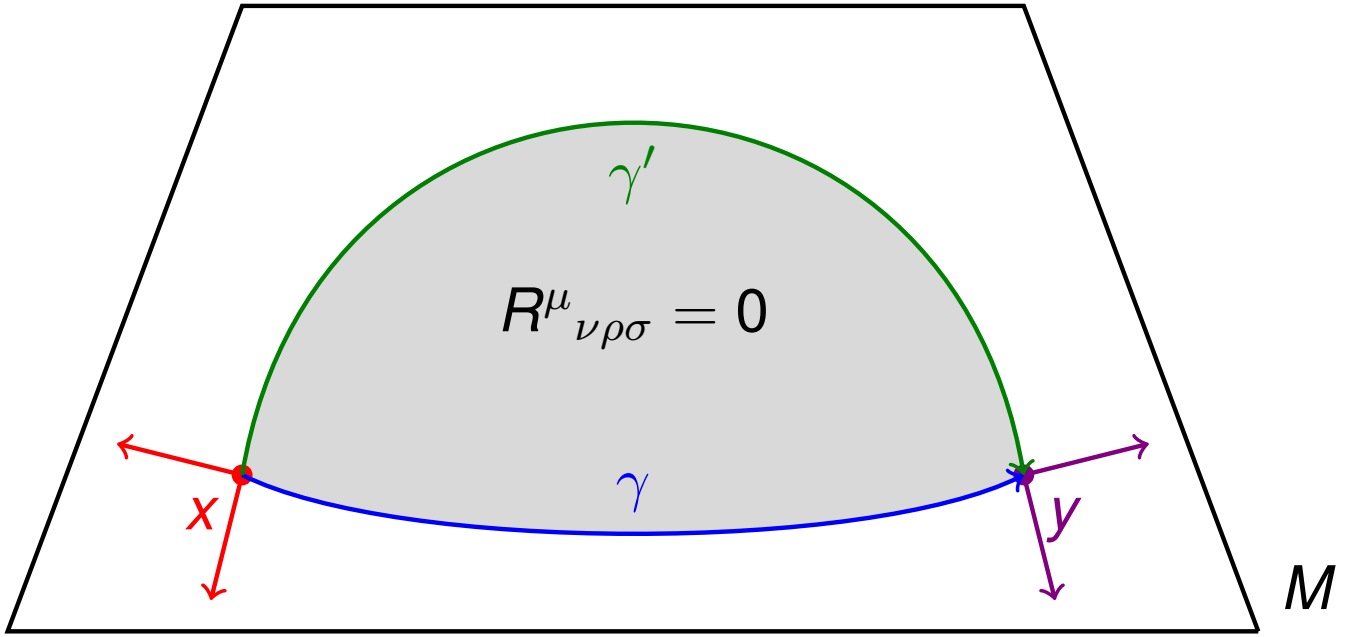
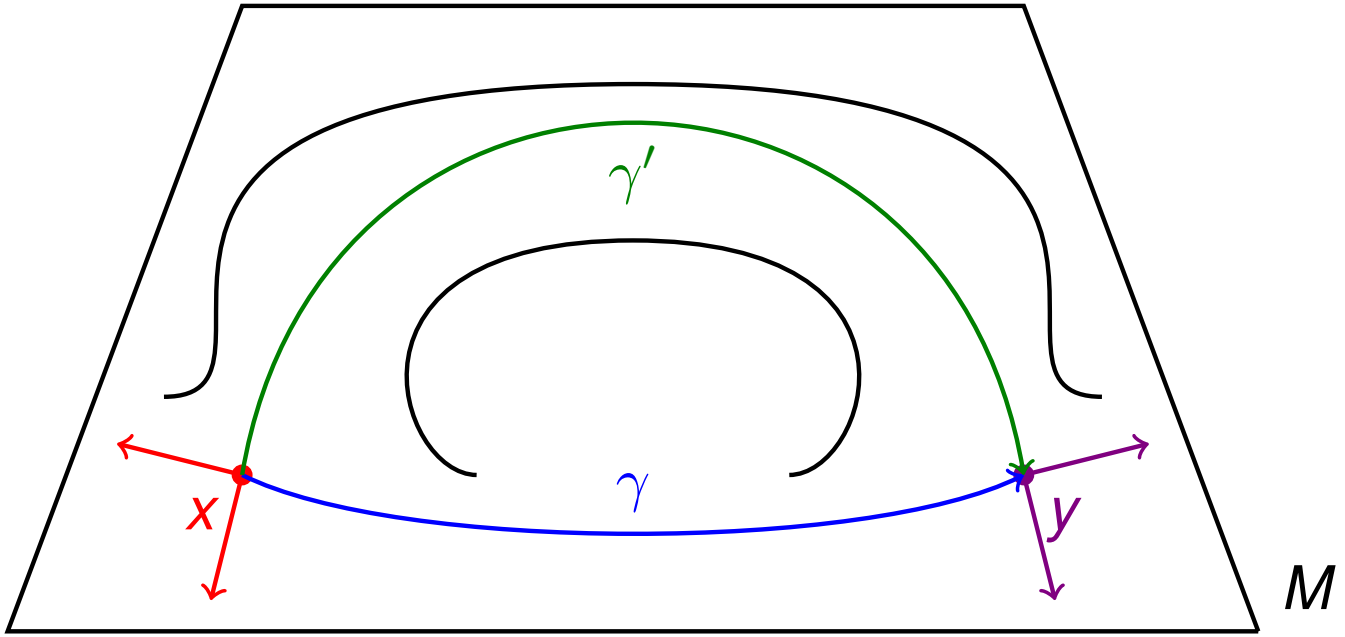


Can we always use the Weitzenböck gauge?

- Recipe for integrating the connection:
  1. At some $x \in M$, choose $\overset{w}{\theta}{}^a{}_\mu(x)$ to fit with the metric.
  2. For any other $y \in M$, choose path $x \overset{\gamma}{\rightsquigarrow} y$, and parallel transport.
- What happens if we choose another path $x \overset{\gamma'}{\rightsquigarrow} y$?
  - ✓ Vanishing curvature: parallel transport along both path agrees.
  - ↯ But only if γ and γ' are homotopic paths!



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- Consider local Lorentz transformations $\Lambda : M \to \mathrm{SO}(1,3)$:
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$\Rightarrow$ The "usual rules" for playing with "dark" fields apply:
  - Find out which degrees of freedom couple to physical observables.
  - "Remnant symmetries" may yield gauge degrees of freedom.
  - Make sure physical degrees of freedom obey healthy evolution.
  - ↯ Pay attention to possible pathologies:
    - Is the evolution of physical degrees of freedom determined?
    - Are the physical degrees of freedom stable under perturbations?[1]
    - Does the theory remain healthy under quantization?

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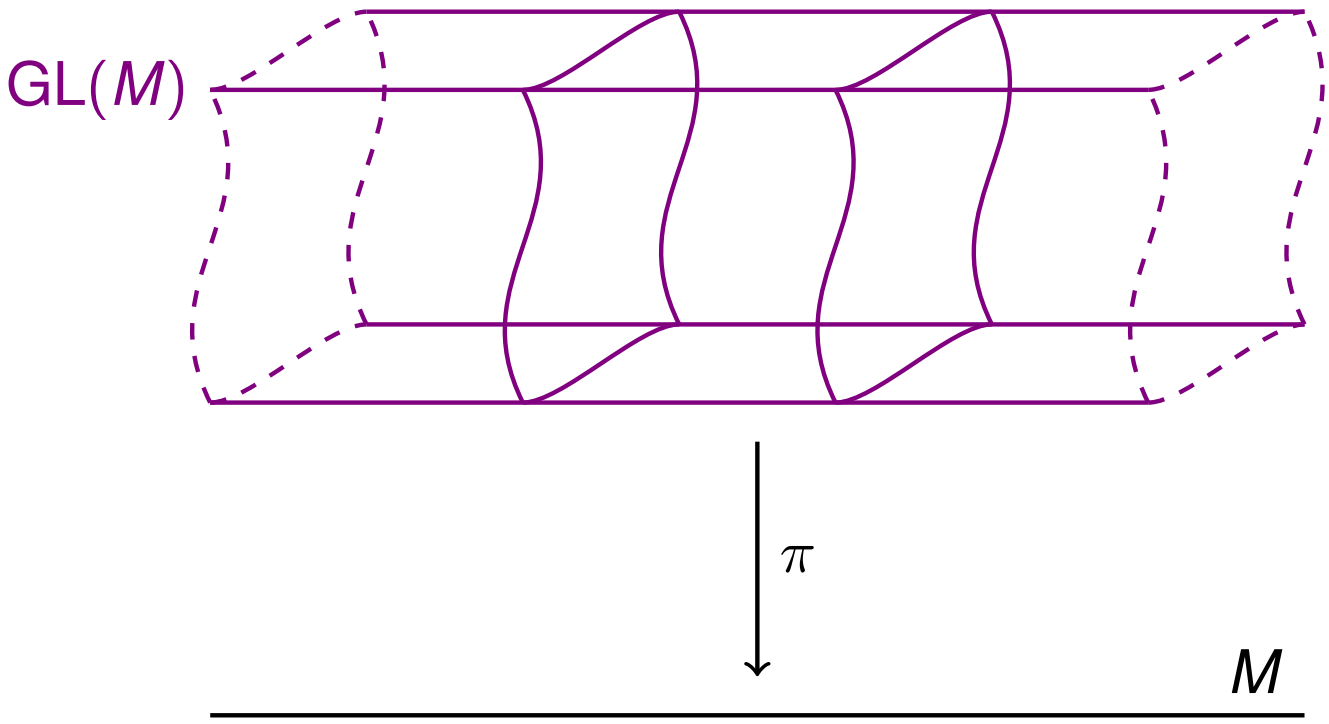
$\Rightarrow$ Most fundamental variables found in geometric picture.

The geometric picture

1. Start with the general linear frame bundle $\pi : \mathrm{GL}(M) \to M$.



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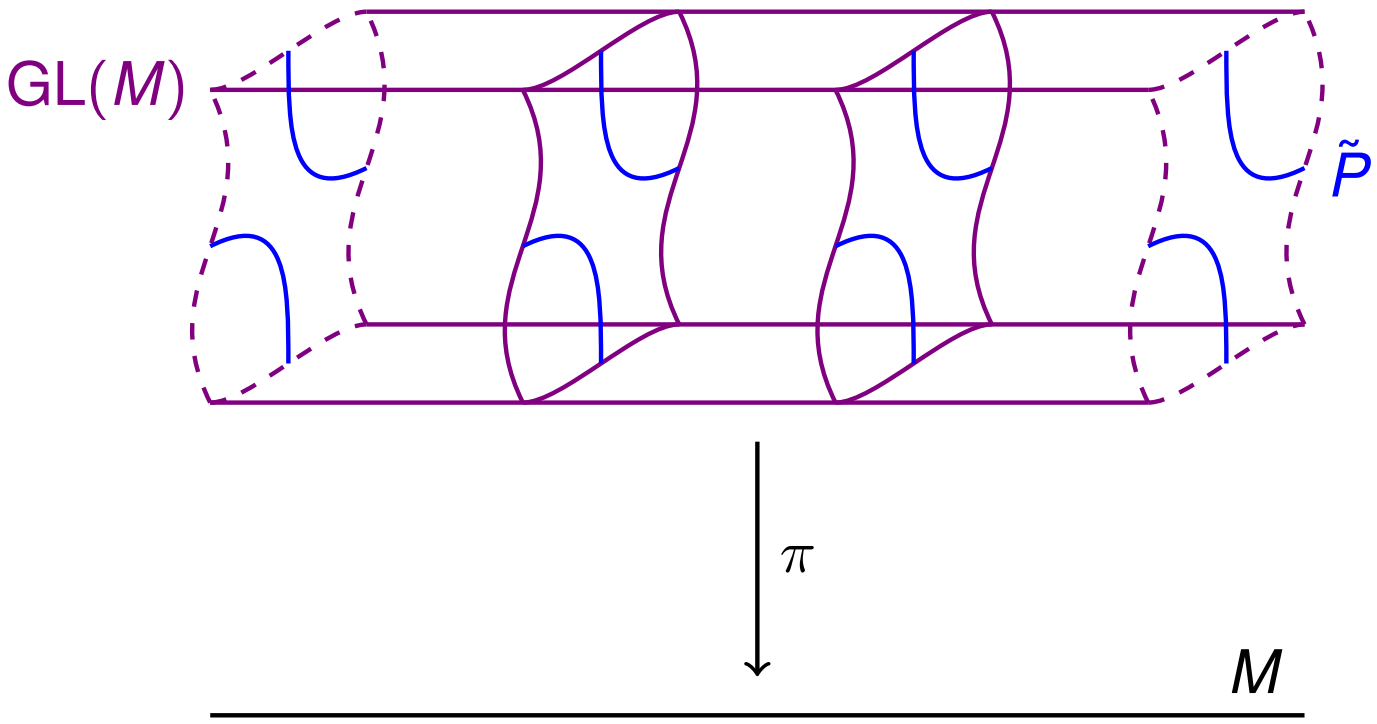

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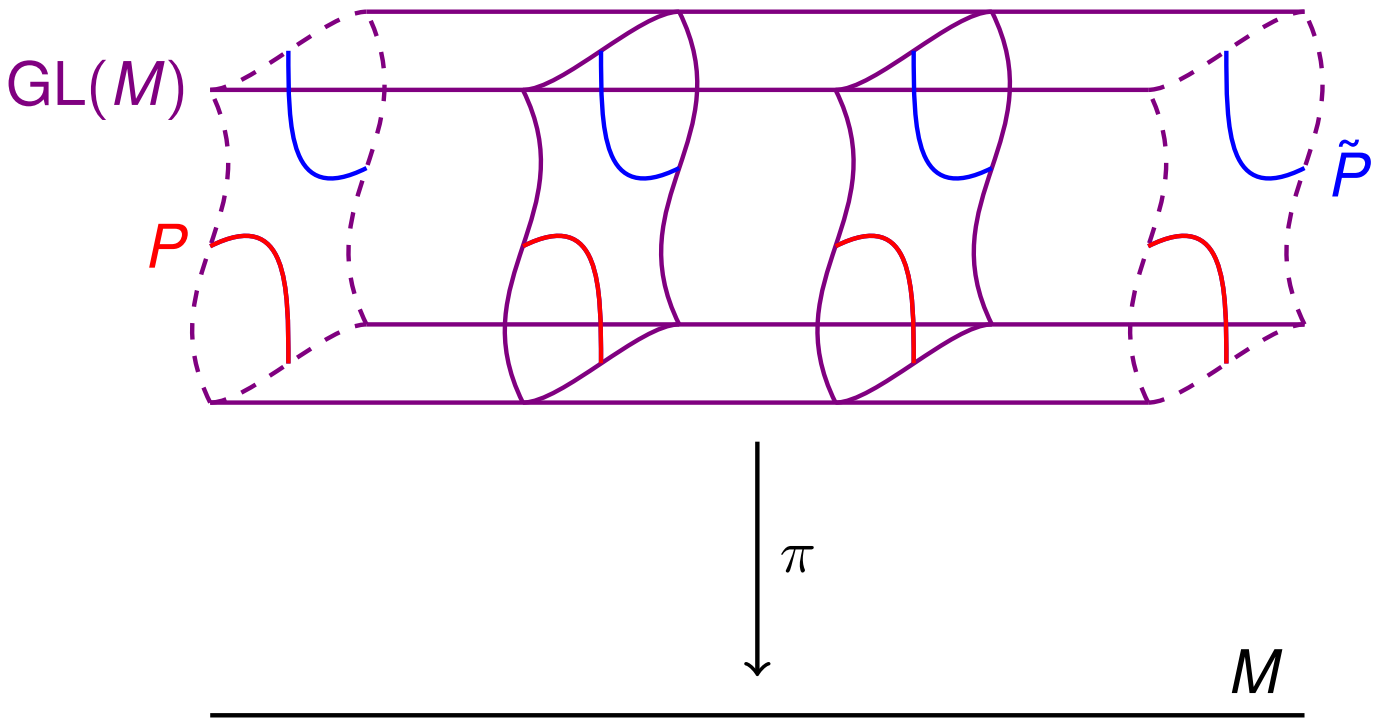

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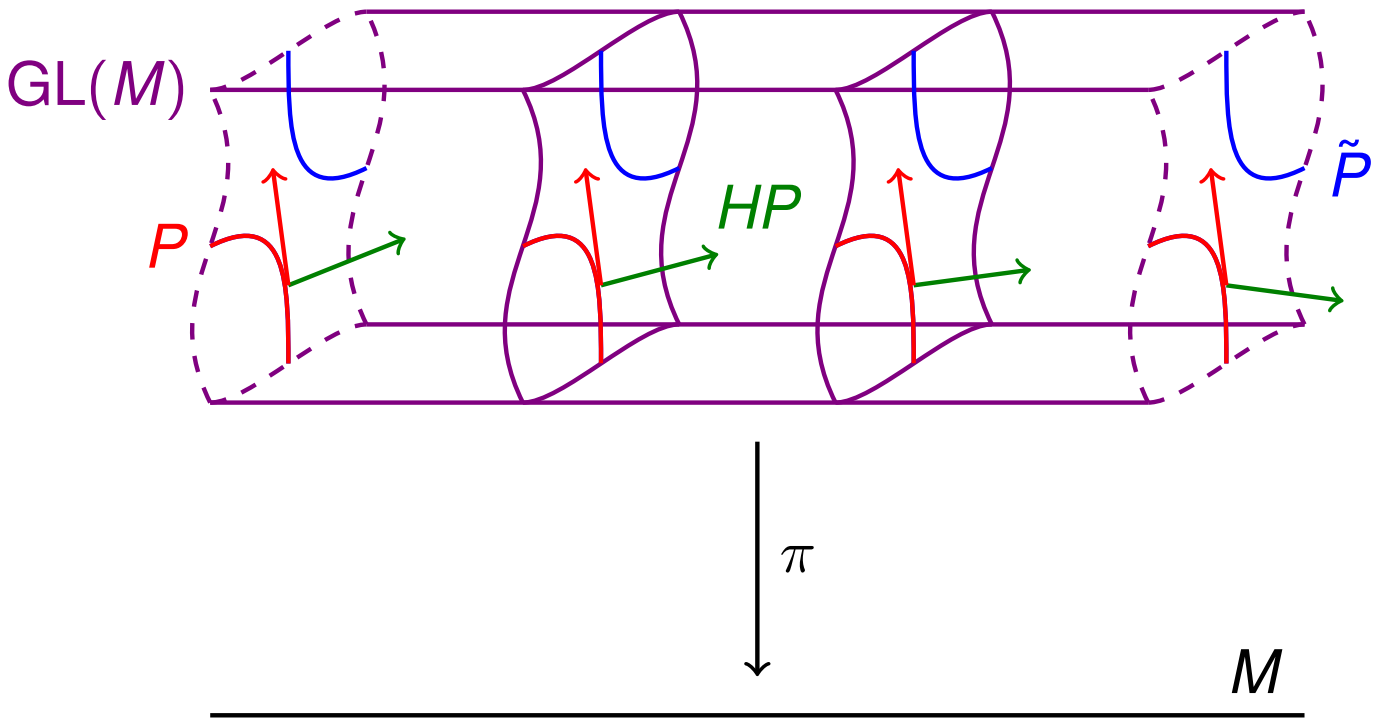

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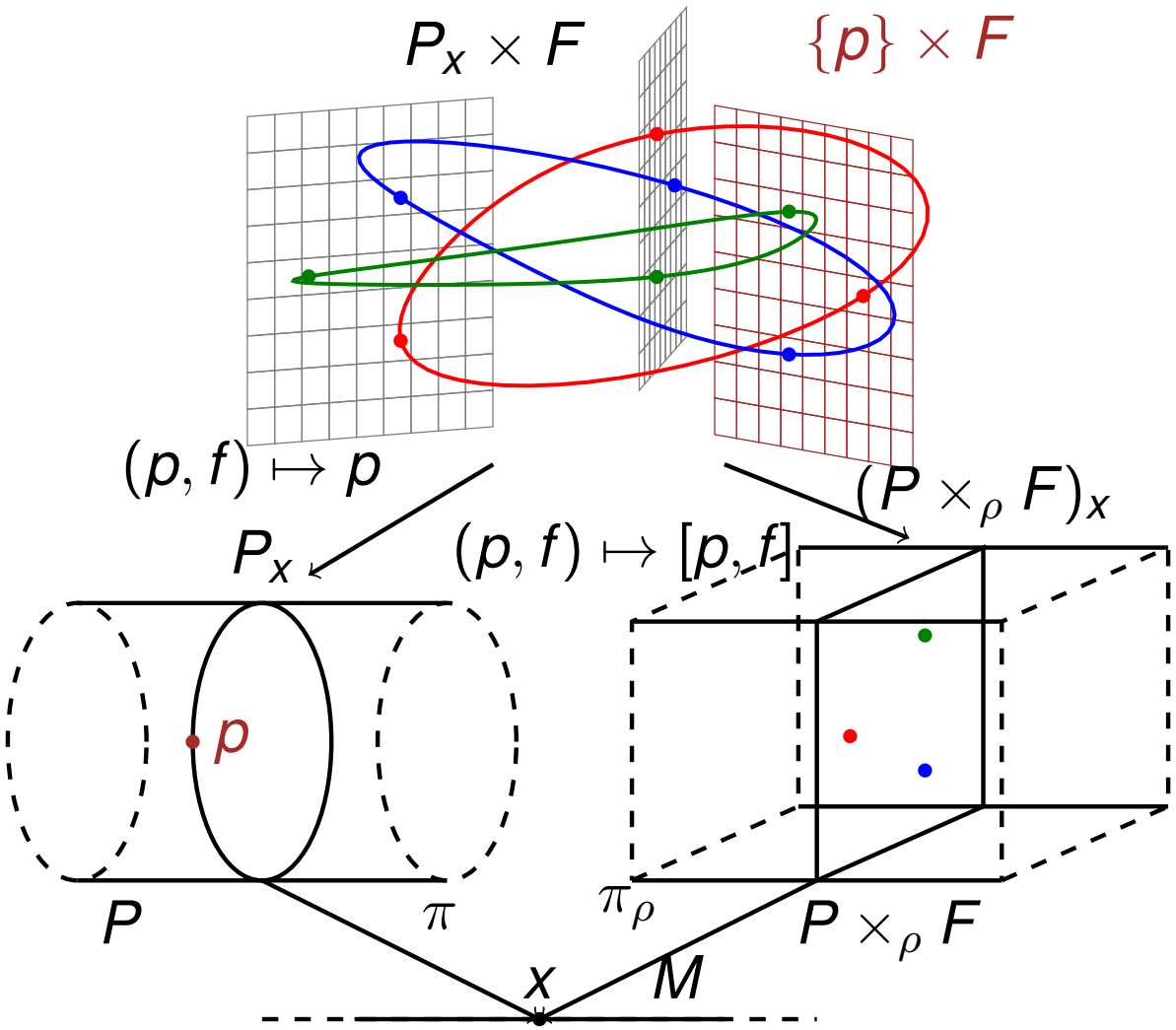
Mantra

In order to understand gravity, one must understand geometry.

Extra: the associated bundle



Extra: the many faces of connections